Vectors and Tensors:

(Read Ch. 1.7 for refresher on vectors).

Vectors: \( \vec{A} \) has magnitude \( |\vec{A}| \) and direction \( \theta \).
A vector is also a tensor of rank 1 characterized as $A_i$.

$i$ runs from 1 to $N$ where $N$ is the dimension of the space.

All the laws of Physics can be expressed in terms of tensors.
<table>
<thead>
<tr>
<th>rank</th>
<th>notation</th>
<th>name</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>a</td>
<td>scalar</td>
<td>mass, charge, speed, ...</td>
</tr>
<tr>
<td>1</td>
<td>ai</td>
<td>vectors</td>
<td>velocity, force, field,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>moment of inertia, ...</td>
</tr>
<tr>
<td>2</td>
<td>aij</td>
<td>matrix</td>
<td>octopole moment</td>
</tr>
<tr>
<td>3</td>
<td>aijk</td>
<td>cube</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>aijk e</td>
<td>hypercube</td>
<td>stress tensor</td>
</tr>
</tbody>
</table>
Multiple expansion:

\[ q_0 = \int \rho(r) \, dr \]
\[ p_i = \int \rho(r) r_i \, dr \]
\[ q_{ij} = \int \rho(r) r_i r_j \, dr \]
\[ q_{ijk} = \int \rho(r) r_i r_j r_k \, dr \]

Quadrupole

Octopole moment

Stress Tensor:

\[ \varepsilon_{ab} = \sum \frac{\sigma_{abcd} \sigma_{cd}}{c_{ad}} \]

Strain (Tensor of rank 2)

Stress (Tensor of rank 2)
A tensor of rank \( k \) has \( N^k \) components (\( N \): space dimension)

\[ x \times \text{Vectors} \]

\( N = 3 \)

Tensor notation:
\[
\begin{align*}
x &= 1 \\
y &= 2 \\
z &= 3
\end{align*}
\]

\[ |\vec{A}| : \text{magnitude} \]
\[ \sqrt{A_x^2 + A_y^2 + A_z^2} \]

Scalar

\[ A_x = |\vec{A}| \cos \alpha \]
\[ A_y = |\vec{A}| \cos \beta \]
\[ A_z = |\vec{A}| \cos \gamma \]

director cosines
\[ \cos^2 x + \cos^2 y + \cos^2 z = 1 \quad \text{the #1.} \]
Systems of Reference

Cartesian System

$A_1$ is the projection parallel to $x_2$ or perpendicular to $x_1$.

\[ \vec{\mathbf{A}} = (A_1, A_2) = A_2 \hat{e}_2 = A_1 \hat{e}_1 \]

\[ A_1 = |\vec{\mathbf{A}}| \cos \alpha \]

\[ A_2 = |\vec{\mathbf{A}}| \cos \beta \]

\[ |\vec{\mathbf{A}}| = \sqrt{A_1^2 + A_2^2} \]

\[ \cos^2 \alpha + \cos^2 \beta = 1 \]

Oblique System (crystalline)

\[ \vec{\mathbf{A}} = (A_1', A_2') = A_2' \hat{e}_2' = A_1' \hat{e}_1' \]

\[ \vec{\mathbf{A}} = (A_1, A_2) = A_i \hat{e}_i \] (parallel projection)

\[ \vec{\mathbf{A}} = (A_1', A_2') = A_i' \hat{e}_i' \] (perpendicular projection)

Two sets of components.

$A_i$: covariant

$A_i'$: contravariant

Later to be shown.
General definition of vector

A vector is an entity whose coordinates transform in a well-defined way from system $k$ to $k'$. The transformation is given by the behavior of a "prototype" vector.

Prototype: vector position $\mathbf{r} \in \mathbb{R}^3$
Prototype vector $\mathbf{r}$: $\mathbf{r} = (x_1, x_2)$

Find:

$x_1' = f(x_1, x_2)$

$x_2' = f(x_1, x_2)$

$x_1 = r \cos \theta$

$x_1' = r \cos (\theta - \gamma) = r \cos \theta \cos \gamma + r \sin \theta \sin \gamma = x_1 \cos \gamma + x_2 \sin \gamma$

$x_2 = r \sin \theta$

$x_2' = r \sin (\theta - \gamma) = r \sin \theta \cos \gamma - r \cos \theta \sin \gamma = -x_1 \sin \gamma + x_2 \cos \gamma$
Then we can write:

\[
\begin{pmatrix}
    x_1' \\
    x_2'
\end{pmatrix} = \begin{pmatrix}
    \cos \gamma & \sin \gamma \\
    -\sin \gamma & \cos \gamma
\end{pmatrix} \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

or

\[
x_2' = \sum_{j=1}^{2} M_{ij} x_j \equiv M_{ij} x_j
\]

Einstein's notation
Sum over repeated indices.