SOLUTION:

a) We need to express the coordinates of \( \mathbf{r}' \) in \( S' \) in terms of the coordinates of \( \mathbf{r} \) in \( S \). We see that the \( z \) component of \( \mathbf{r} \) is not affected by the change of system. Thus we need to concentrate on the components of \( \mathbf{r} \) parallel to the plane perpendicular to \( z = x_3 \) as shown in the figure:

\[
X_2 = X'_2 \quad X^2 = X'^2 \quad \alpha
\]

\[
\mathbf{r}_\parallel = \mathbf{r}'_\parallel
\]

\[X'_1 = X'_1 \quad X^1 = X'^1 \quad \alpha
\]

**FIG. 1:** Decomposition of the parallel component of \( \mathbf{r} \) in \( S \) and \( S' \).

We see that

\[
x'^1 \cos \alpha = x^1, \quad \text{(1)}
\]

\[
x^2 - x'^2 = x'^1 \sin \alpha. \quad \text{(2)}
\]

Then, solving for \( x'^1 \) in Eq.(1) we obtain:

\[
x'^1 = \sec \alpha x^1, \quad \text{(3)}
\]

and plugging Eq.(3) in Eq.(2) and solving for \( x'^2 \) we obtain:

\[
x'^2 = x^2 - \tan \alpha x^1, \quad \text{(4)}
\]

and

\[
x'^3 = x^3. \quad \text{(5)}
\]

b) We can obtain \( M^i_j = \partial x'^i / \partial x^j \) from Eqs. (3,4,5):

\[
M^i_j = \begin{bmatrix}
\sec \alpha & 0 & 0 \\
-\tan \alpha & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

b) We can obtain \( M^i_j = \partial x'^i / \partial x^j \) from Eqs. (3,4,5):

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M^i_j = \begin{bmatrix}
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\( \hat{\mathbf{e}}_2 = (0, 1, 0), \)  
\( \hat{\mathbf{e}}_3 = (0, 0, 1). \)  

\[
\hat{\mathbf{e}}_i = \frac{\partial x'^j}{\partial x^i} \mathbf{e}'_j, \tag{10}
\]

We need to solve the system of equations given by:

\( \hat{\mathbf{e}}_1 = \sec \alpha \mathbf{e}'_1 - \tan \alpha \mathbf{e}'_2, \tag{11} \)

\( \hat{\mathbf{e}}_2 = \mathbf{e}'_2, \tag{12} \)

and

\( \hat{\mathbf{e}}_3 = \mathbf{e}'_3. \tag{12} \)

We obtain:

\( \mathbf{e}'_1 = \cos \alpha \mathbf{e}_1 + \sin \alpha \mathbf{e}_2, \tag{13} \)

\( \mathbf{e}'_2 = \mathbf{e}_2, \tag{14} \)

and

\( \mathbf{e}'_3 = \mathbf{e}_3. \tag{15} \)

e) We know that

\[
g'_{ij} = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{pmatrix}
1 & \sin \alpha & 0 \\
\sin \alpha & 1 & 0 \\
0 & 0 & 1 
\end{pmatrix}. \tag{16}
\]

f) We know that

\( x'_i = g'_{ij} x'^j, \tag{17} \)

and in part (a) we obtained \( x'^j \) in terms of \( x^i \) then we obtain Then,

\( x'_1 = x^1 \cos \alpha + x^2 \sin \alpha, \tag{18} \)

\( x'_2 = x^2, \tag{19} \)

and

\( x'_3 = x^3. \)  

g) We know that in $S \mathbf{r}=(x^1, x^2, x^3) = (3, 2, 1)$ then from (a) we find that

$$x^1 = 3 \sqrt{2},$$

$$x^2 = -1,$$  \hspace{1cm} (21)

and

$$x^3 = 1.$$ \hspace{1cm} (23)

The relationship between $x^i$ and $x'_j$ was obtained in (f) then using these results we obtain:

$$x'_1 = 5 \sqrt{2},$$  \hspace{1cm} (24)

$$x'_2 = 2,$$  \hspace{1cm} (25)

and

$$x'_3 = 1.$$  \hspace{1cm} (26)

h) The results obtained in (g) allow to calculate

$$x'_i x'^i = 15 - 2 + 1 = 14.$$  \hspace{1cm} (27)

In $S$ we obtain

$$x_i x^i = 9 + 4 + 1 = 14.$$  \hspace{1cm} (28)

As expected the results are equal in both systems,

i) The tensor $T^{nij}_{kl}$ has rank 4 because it has 4 indices. Since we are in three dimensions the tensor has $3^4 = 81$ components.

j) We need to use the transformation rules to go from $S'$ to $S$ for each index then:

$$T'^{mn}_{op} = \frac{\partial x^m}{\partial x'^n} \frac{\partial x'^k}{\partial x^o} \frac{\partial x'^l}{\partial x^p} T^{nij}_{kl}. \hspace{1cm} (29)$$

k) I know that

$$T^{112}_{21} = x^1 x^2 x'_2 x'_1 = 3 \sqrt{2} \times (-1) \times 2 \times 5 \sqrt{2}/2 = -30,$$  \hspace{1cm} (30)

while

$$T^{112}_{12} = x^1 x^2 x_2 x_1 = 3 \times 2 \times 2 \times 3 = 36.$$ \hspace{1cm} (31)

Since they are the components of a tensor we do not expect the two to be equal.

l) We see that $T'^{mnop}$ is a tensor symmetric under the exchange of any pair of indices. This means that of its $3 \times 3 \times 3 \times 3 = 81$ components only 15 of them are independent. We see that the independent components can be counted in the following way:
i) Components with all the indices equal, i.e., iii: 3
ii) Components with 3 equal indices and one different, iiij: 6 independent (3 values for i and 2 values left for j).
iii) Components with indices equal in pairs, iiij: 3 independent.
iv) Components with 2 equal indices and the other 2 different, iijk: 3 independent.

m) The tensor $T^{ijij}$ has rank 0 because all its indices are contracted, i.e., it is a scalar.

n) We know that
\[ T^{ijij} = r^i r^j r'_i r'_j = x^i x'_i x^j x'_j = 14 \times 14 = 196, \]  
(32)
where I used the result of part (h) Eq.(27).

o) We know that
\[ T^{ijij} = r^i r^j r_j r_i = x^i x_i x^j x_j = 14 \times 14 = 196, \]  
(33)
where I used the result of part (h) Eq.(28). We see that $T = T'$ as it should be since the tensor is a scalar.