$$H = \hbar\omega\left(a_- a_+ - \frac{1}{2}\right)$$

Or, using the other order, we get (left as exercise):

$$H = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)$$

Theorem: if ψ satisfies $H\psi = E\psi$, then $H(a_+\psi) = (E + \hbar\omega)(a_+\psi)$

$$H(a_+\psi) = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)(a_+\psi) = \hbar\omega\left(a_+a_-a_+ + \frac{1}{2}a_+\right)\psi$$

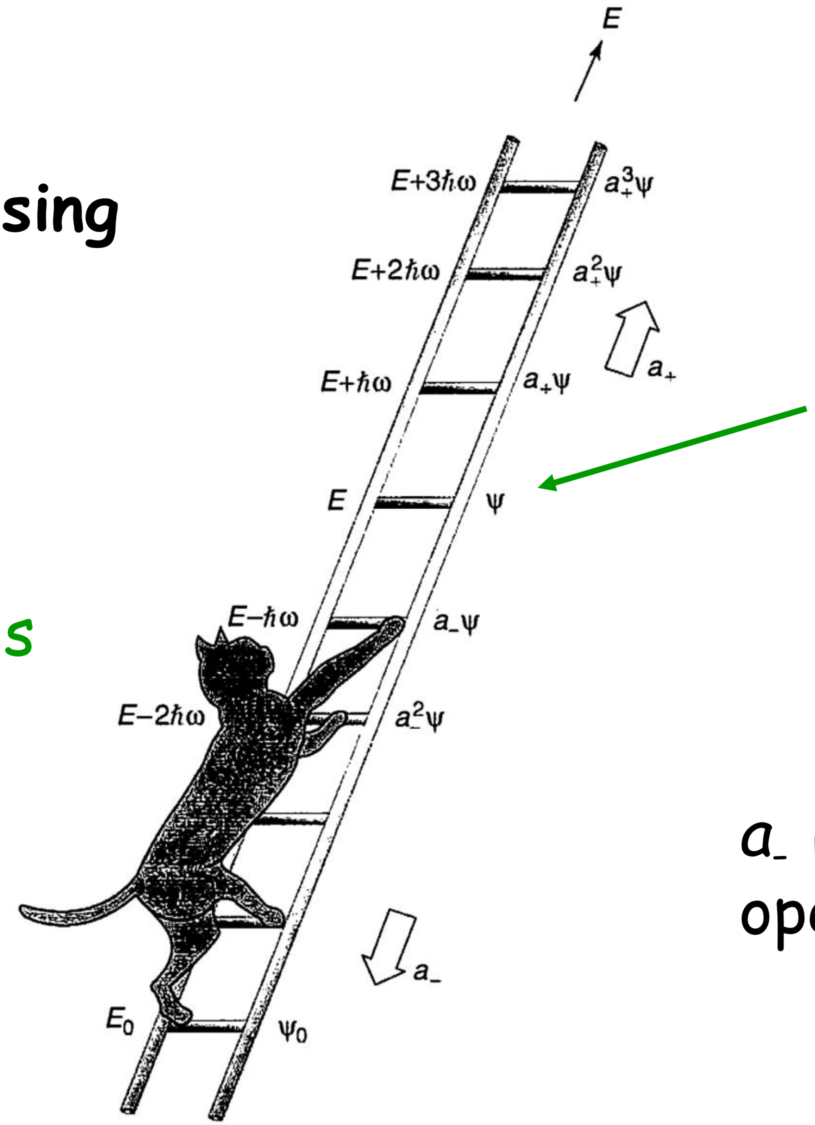
$$= \hbar\omega a_+\left(a_-a_+ + \frac{1}{2}\right)\psi = a_+\left[\hbar\omega\left(a_-a_+ + 1 - \frac{1}{2}\right)\psi\right]$$

$$= a_+(H + \hbar\omega)\psi = a_+(E + \hbar\omega)\psi = (E + \hbar\omega)(a_+\psi).$$

If I know one solution, I know another solution …

Another theorem: if ψ satisfies $H\psi = E\psi$, then $H(a_-\psi) = (E - \hbar\omega)(a_-\psi)$ *(left as exercise)*

a_+ is the **raising** operator



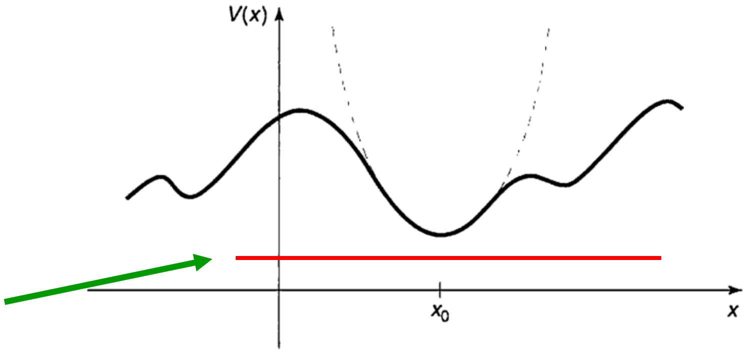
If I am given one solution, I get infinite more.

Equally spaced energies. This is why it works.

Not general: valid for oscillators.

a_- is the **lowering** operator

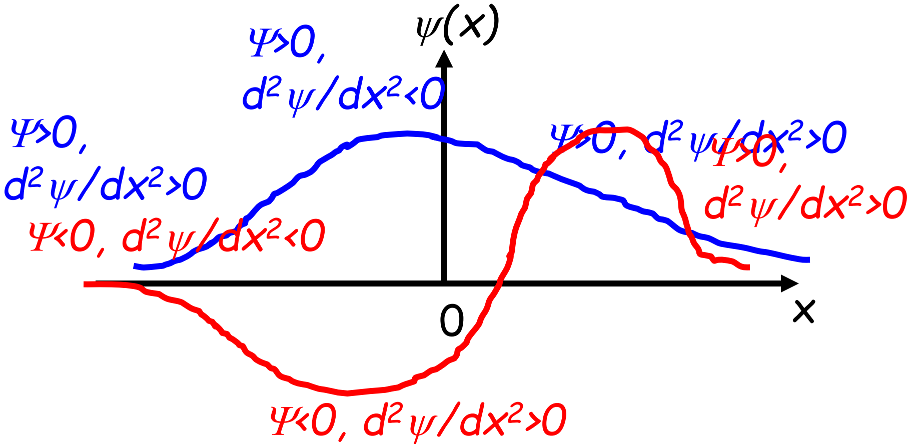
However, there is a problem:
the energy cannot continue going down!



Theorem: E less than V(x) cannot happen

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V\psi=E\psi \longrightarrow \frac{d^2\psi}{dx^2}=\frac{2m}{\hbar^2}\underbrace{[V(x)-E]}\psi$$

If this is >0 for all x, then ψ and $d^2\psi/dx^2$ must have same sign.



Because the energy cannot continue going down forever, the chain down must stop…

Eventually once true ground state ψ_0 is reached, then $a_-\psi_0$ must be 0. We can use this condition to find ψ_0.

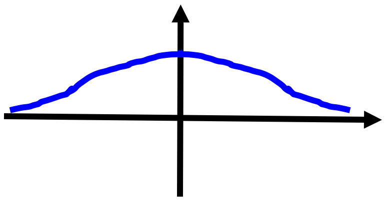
$$\underbrace{\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx}+m\omega x\right)}_{a_-}\psi_0=0 \longrightarrow \frac{d\psi_0}{dx}=-\frac{m\omega}{\hbar}x\psi_0$$

$$\int\frac{d\psi_0}{\psi_0}=-\frac{m\omega}{\hbar}\int x\,dx \quad\Rightarrow\quad \ln\psi_0=-\frac{m\omega}{2\hbar}x^2+\text{constant},$$

$$\boxed{\psi_0(x)=Ae^{-\frac{m\omega}{2\hbar}x^2}.}$$

After normalization:
(left as exercise)

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$



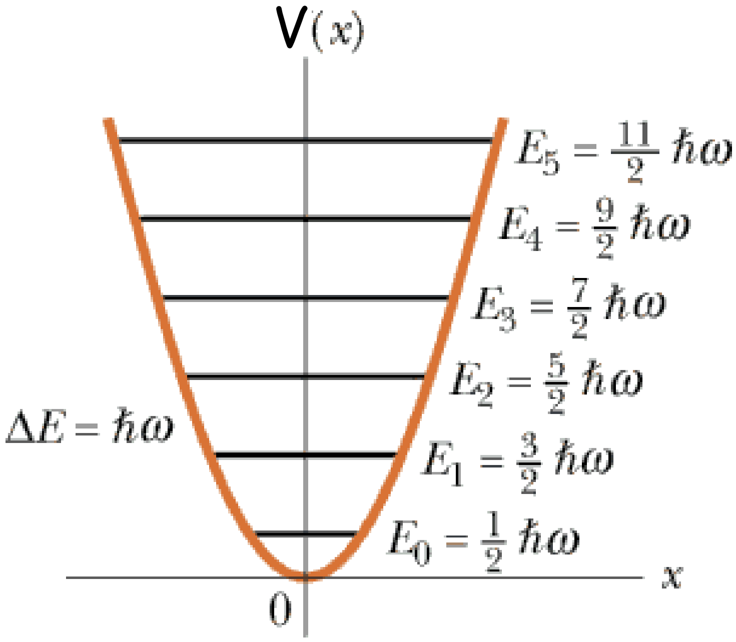
What is the energy E_0?

=0

$$\underbrace{\hbar\omega(a_+a_- + 1/2)}_{H\text{ (Hamiltonian)}}\psi_0 = E_0\psi_0$$

Then: $$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

>0 as expected.
"Zero point energy".
The harmonic oscillators are never still!



Equally spaced levels

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega}$$

Solutions are

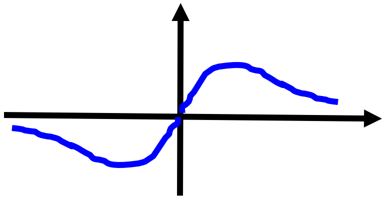
$$\boxed{\psi_n(x) = A_n (a_+)^n \psi_0(x)}$$

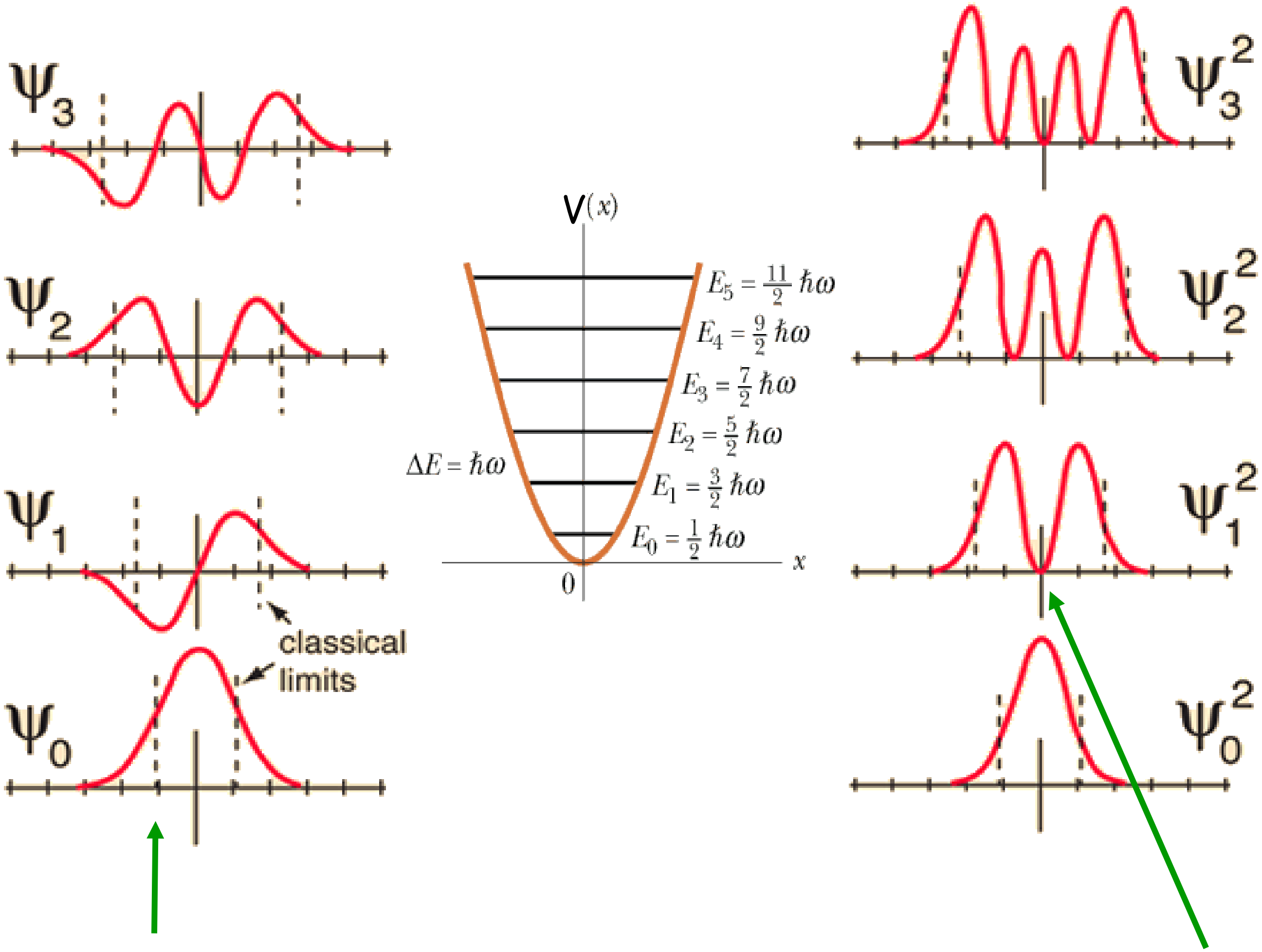
Example: construct state 1

Brings down an *x*

$$\psi_1(x) = A_1 a_+ \psi_0 = \frac{A_1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= A_1\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\,x e^{-\frac{m\omega}{2\hbar}x^2}.$$





Note: particle can be found outside the classical region

Zero chance at the nodes.

It can be shown, following the textbook, that:

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$\psi_n = \frac{1}{\underbrace{\sqrt{n!}}_{A_n}}(a_+)^n\psi_0$$

$$\int_{-\infty}^{\infty}\psi_m^*\psi_n\,dx = \delta_{mn}$$

Orthonormal like we found
before for square well.

Example 2.5: find expectation value of V in the nth state.

$$\langle V\rangle = \left\langle \frac{1}{2}m\omega^2 x^2 \right\rangle = \frac{1}{2}m\omega^2 \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n \, dx$$

It may be tempting to write ψ_n as a Gaussian with some polynomial in front. However, there is a simpler path.

From $a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$ used before, deduce:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-); \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$x^2 = \frac{\hbar}{2m\omega}\left[(a_+)^2 + (a_+a_-) + (a_-a_+) + (a_-)^2\right]$$

Creates ψ_{n-2} orthog to ψ_n

$$\langle V\rangle = \frac{\hbar\omega}{4}\int \psi_n^*\left[(a_+)^2 + (a_+a_-) + (a_-a_+) + (a_-)^2\right]\psi_n\,dx.$$

Creates ψ_{n+2} orthog to ψ_n

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \qquad a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

$$a_-a_+\psi_n = a_-\sqrt{n+1}\,\psi_{n+1} = \sqrt{n+1}\,\underbrace{a_-\psi_{n+1}}_{\sqrt{n+1}\,\psi_n} = (n+1)\psi_n$$

$$\boxed{\langle V\rangle = \frac{\hbar\omega}{4}(\overset{a_+a_-}{n} + \overset{a_-a_+}{n+1}) = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)} = \frac{1}{2}E_n$$