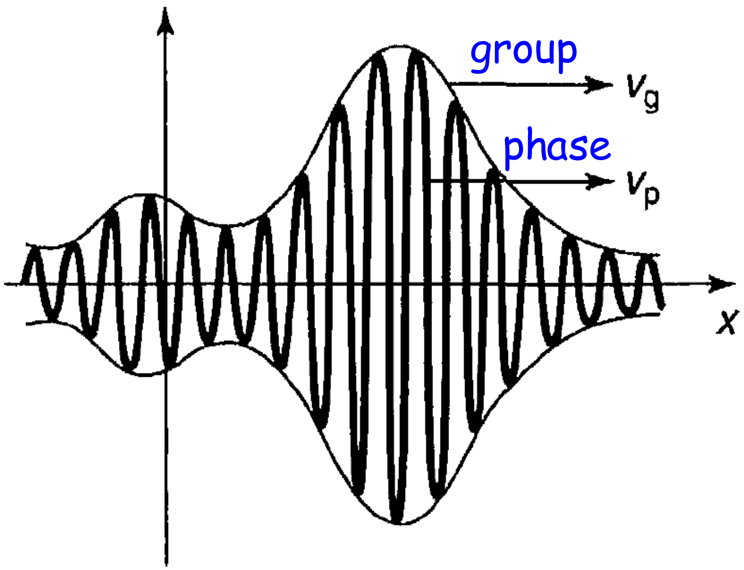
The paradox $v=(1/2)\,p/m$ can also be addressed in the context of wave packets. The speed of the sinusoidal components (phase velocity) is not important. The envelope's speed (group velocity) is physically relevant.




$$\omega = (\hbar k^2/2m)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx-\omega t)}\, dk.$$

Assume $\phi(k)$ is narrowly peaked $k=k_0+s$. Thus, $k^2 \sim k_0^2 + 2k_0 s$.

$\hbar k_0^2/2m$ $\hbar k_0/m$

$$\Psi(x,t) \cong \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k_0+s) e^{i[(k_0+s)x-(\omega_0+\omega_0' s)t]}\, ds$$

Drop $s^2$ term

$$\Psi(x,t) \cong \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k_0+s) e^{i[(k_0+s)x-(\omega_0+\omega_0' s)t]}\, ds$$

$-\omega_0' st = -\omega_0' st + k_0\omega_0' t - k_0\omega_0' t =$

$= + k_0\omega_0' t - (k_0+s)\,\omega_0' t$

Then

$$\Psi(x,t) \cong \frac{1}{\sqrt{2\pi}} e^{i(-\omega_0 t + k_0\omega_0' t)} \int_{-\infty}^{+\infty} \phi(k_0+s) e^{i(k_0+s)(x-\omega_0' t)}\, ds$$

Since

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k_0+s) e^{i(k_0+s)x}\, ds$$

then

$$\Psi(x,t) \cong e^{-i(\omega_0 - k_0\omega_0')t}\, \Psi(x-\omega_0' t, 0)$$

$\omega_0' = \hbar k_0/m = p_0/m$

Not important for $|\Psi|^2$

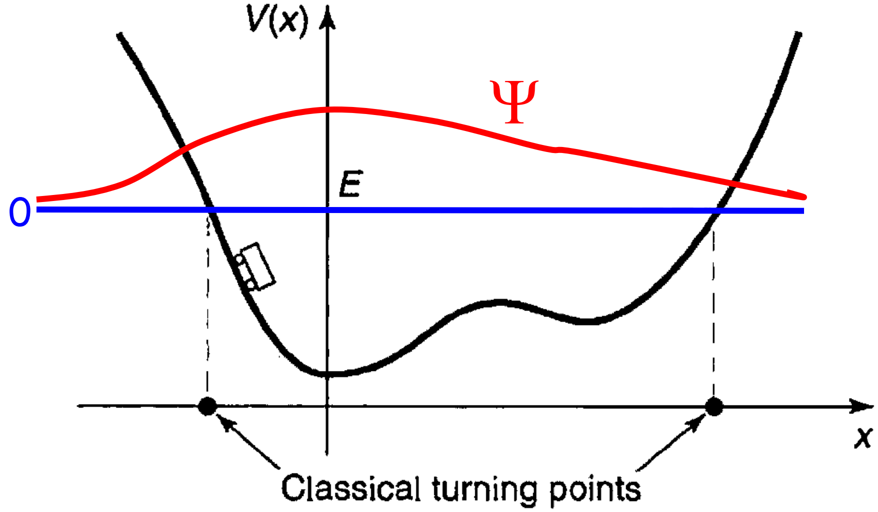
Same as original wave packet but moving!

velocity of wave packet

Velocity of wave packet is correctly $v_{classical}$ !!

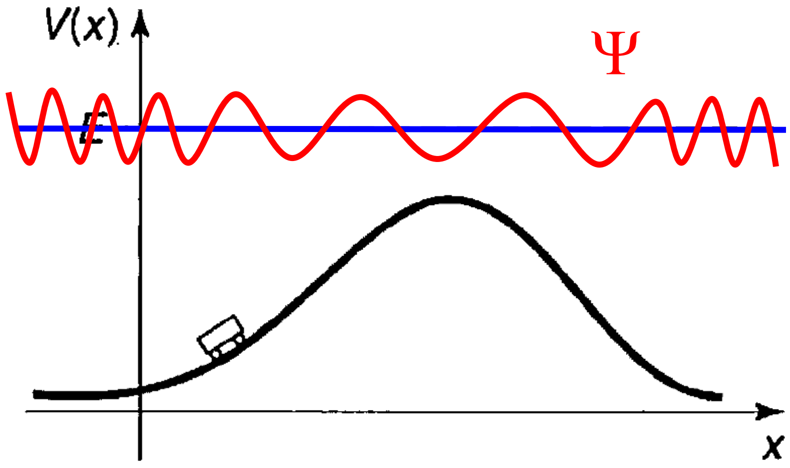
# Bound vs scattering states

In V(x) below, where E< V(+-infinity), classically the particle oscillates back and forth and cannot escape. Any E is good.




Bound state

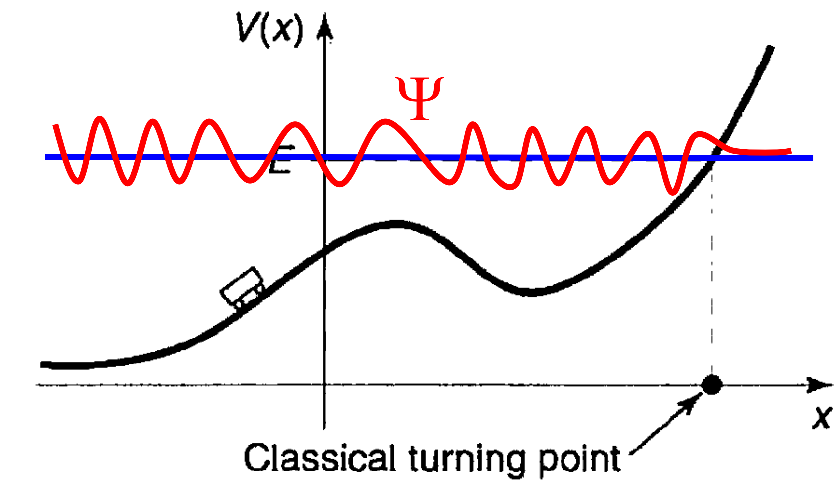
In QM, E is discrete (index n) and Ψ is normalizable.
Particle still cannot escape because Ψ → +- infinite..




E is a "scattering state" both in QM and classically, because E>V(+- infinity).

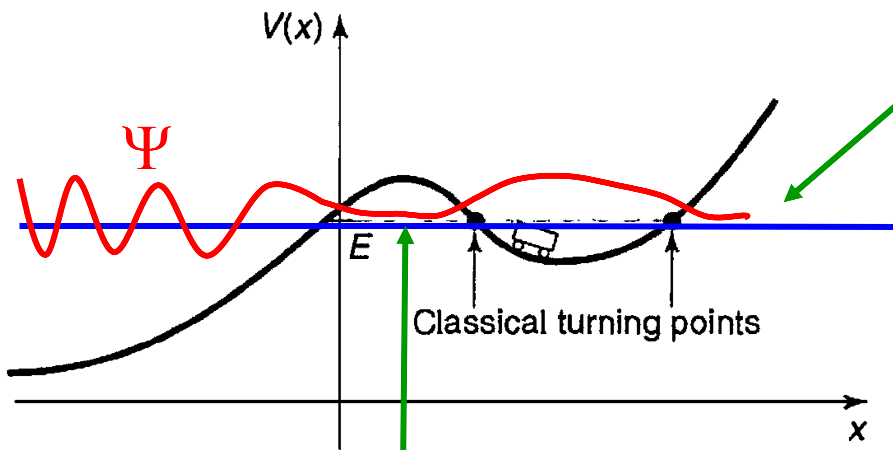
In both, E is continuous. In QM, solutions are not normalizable, but wave packets work.

Students must develop the ability to sketch the wave function even without solving the problem (most problems cannot be solved exactly)




Scattering state both in QM and classically for this E.

In QM, there is a small penetration beyond classical point.




Classically this E can be either bound or scattering state depending on initial location at t=0.

In QM, it is only one scattering state. Particle can escape!

Reason: tunneling.