(In Special Relativity this is the trajectory of a particle subject to a constant force $F = mc^2/b$.) Sketch the graph of w versus t. At four or five representative points on the curve, draw the trajectory of a light signal emitted by the particle at that point—both in the plus x direction and in the minus x direction. What region on your graph corresponds to points and times (x, t) from which the particle cannot be seen? At what time does someone at point x first see the particle? (Prior to this the potential at x is evidently zero.) Is it possible for a particle, once seen, to *disappear* from view?

! **Problem 10.16** Determine the Liénard-Wiechert potentials for a charge in hyperbolic motion (Eq. 10.45). Assume the point $\mathbf{r}$ is on the x axis and to the right of the charge.

10.3.2 The Fields of a Moving Point Charge

We are now in a position to calculate the electric and magnetic fields of a point charge in arbitrary motion, using the Liénard-Wiechert potentials:[11]

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2}V(\mathbf{r}, t), \tag{10.46}$$

and the equations for **E** and **B**:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The differentiation is tricky, however, because

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \quad \text{and} \quad \mathbf{v} = \dot{\mathbf{w}}(t_r) \tag{10.47}$$

are both evaluated at the retarded time, and t_r—defined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r) \tag{10.48}$$

—is *itself* a function of $\mathbf{r}$ and t.[12] So hang on: the next two pages are rough going ... but the answer is worth the effort.

Let's begin with the gradient of V:

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{-1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\nabla(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}). \tag{10.49}$$

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[11] You can get the fields directly from Jefimenko's equations, but it's not easy. See, for example, M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed., Sect. 8.4 (Orlando, FL: Saunders, 1995).

[12] The following calculation is done by the most direct, "brute force" method. For a more clever and efficient approach see J. D. Jackson, *Classical Electrodynamics*, 3d ed., Sect. 14.1 (New York: John Wiley, 1999).

Since $\mathscr{r} = c(t - t_r)$,

$$\nabla \mathscr{r} = -c \nabla t_r. \tag{10.50}$$

As for the second term, product rule 4 gives

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}}). \tag{10.51}$$

Evaluating these terms one at a time:

$$\begin{aligned}(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} &= \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right)\mathbf{v}(t_r) \\ &= \mathscr{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z} \\ &= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r),\end{aligned} \tag{10.52}$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the *acceleration* of the particle at the retarded time. Now

$$(\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} = (\mathbf{v} \cdot \nabla)\mathbf{r} - (\mathbf{v} \cdot \nabla)\mathbf{w}, \tag{10.53}$$

and

$$\begin{aligned}(\mathbf{v} \cdot \nabla)\mathbf{r} &= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \\ &= v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} = \mathbf{v},\end{aligned} \tag{10.54}$$

while

$$(\mathbf{v} \cdot \nabla)\mathbf{w} = \mathbf{v}(\mathbf{v} \cdot \nabla t_r)$$

(same reasoning as Eq. 10.52). Moving on to the third term in Eq. 10.51,

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{\mathbf{z}}$$

$$= -\mathbf{a} \times \nabla t_r. \tag{10.55}$$

Finally,

$$\nabla \times \boldsymbol{\mathscr{r}} = \nabla \times \mathbf{r} - \nabla \times \mathbf{w}, \tag{10.56}$$

but $\nabla \times \mathbf{r} = 0$, while, by the same argument as Eq. 10.55,

$$\nabla \times \mathbf{w} = -\mathbf{v} \times \nabla t_r. \tag{10.57}$$

Putting all this back into Eq. 10.51, and using the "BAC-CAB" rule to reduce the triple cross products,

$$\begin{aligned}\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) &= \mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)+\mathbf{v}-\mathbf{v}(\mathbf{v}\cdot\nabla t_r)-\boldsymbol{\mathscr{r}}\times(\mathbf{a}\times\nabla t_r)+\mathbf{v}\times(\mathbf{v}\times\nabla t_r)\\ &= \mathbf{v}+(\boldsymbol{\mathscr{r}}\cdot\mathbf{a}-v^2)\nabla t_r.\end{aligned} \tag{10.58}$$

Collecting Eqs. 10.50 and 10.58 together, we have

$$\nabla V=\frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\left[\mathbf{v}+(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\nabla t_r\right]. \tag{10.59}$$

To complete the calculation, we need to know ∇t_r. This can be found by taking the gradient of the defining equation (10.48)—which we have already done in Eq. 10.50—and expanding out $\nabla\mathscr{r}$:

$$\begin{aligned}-c\nabla t_r &= \nabla\mathscr{r}=\nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}=\frac{1}{2\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}})\\ &= \frac{1}{\mathscr{r}}[(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}}+\boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}})].\end{aligned} \tag{10.60}$$

But

$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}}=\boldsymbol{\mathscr{r}}-\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)$$

(same idea as Eq. 10.53), while (from Eq. 10.56 and 10.57)

$$\nabla\times\boldsymbol{\mathscr{r}}=(\mathbf{v}\times\nabla t_r).$$

Thus

$$-c\nabla t_r=\frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}}-\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)+\boldsymbol{\mathscr{r}}\times(\mathbf{v}\times\nabla t_r)]=\frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}}-(\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\nabla t_r],$$

and hence

$$\nabla t_r=\frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}. \tag{10.61}$$

Incorporating this result into Eq. 10.59, I conclude that

$$\nabla V=\frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\mathbf{v}-(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\boldsymbol{\mathscr{r}}\right]. \tag{10.62}$$

A similar calculation, which I shall leave for you (Prob. 10.17), yields

$$\begin{aligned}\frac{\partial\mathbf{A}}{\partial t} &= \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\Big[(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})(-\mathbf{v}+\mathscr{r}\mathbf{a}/c)\\ &+ \frac{\mathscr{r}}{c}(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\Big].\end{aligned} \tag{10.63}$$

Combining these results, and introducing the vector

$$\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}, \tag{10.64}$$

I find

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})].} \tag{10.65}$$

Meanwhile,

$$\nabla \times \mathbf{A} = \frac{1}{c^2}\nabla \times (V\mathbf{v}) = \frac{1}{c^2}[V(\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla V)].$$

We have already calculated $\nabla \times \mathbf{v}$ (Eq. 10.55) and ∇V (Eq. 10.62). Putting these together.

$$\nabla \times \mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u}\cdot\boldsymbol{\mathscr{r}})^3}\boldsymbol{\mathscr{r}} \times [(c^2 - v^2)\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}].$$

The quantity in brackets is strikingly similar to the one in Eq. 10.65, which can be written. using the BAC-CAB rule, as $[(c^2 - v^2)\mathbf{u} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}]$; the main difference is that we have $\mathbf{v}$'s instead of $\mathbf{u}$'s in the first two terms. In fact, since it's all crossed into $\boldsymbol{\mathscr{r}}$ *anyway*, we can with impunity *change* these $\mathbf{v}$'s into $-\mathbf{u}$'s; the extra term proportional to $\hat{\boldsymbol{\mathscr{r}}}$ disappears in the cross product. It follows that

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t).} \tag{10.66}$$

Evidently *the magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.*

The first term in $\mathbf{E}$ (the one involving $(c^2 - v^2)\mathbf{u}$) falls off as the inverse *square* of the distance from the particle. If the velocity and acceleration are both zero, this term alone survives and reduces to the old electrostatic result

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.$$

For this reason, the first term in $\mathbf{E}$ is sometimes called the **generalized Coulomb field**. (Because it does not depend on the acceleration, it is also known as the **velocity field.**) The second term (the one involving $\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})$) falls off as the inverse *first* power of $\mathscr{r}$ and is therefore dominant at large distances. As we shall see in Chapter 11, it is this term that is responsible for electromagnetic radiation; accordingly, it is called the **radiation field**—or. since it is proportional to a, the **acceleration field**. The same terminology applies to the magnetic field.

Back in Chapter 2, I commented that if we could only write down the formula for the force one charge exerts on another, we would be done with electrodynamics, in principle. That, together with the superposition principle, would tell us the force exerted on a test charge Q by any configuration whatsoever. Well ... here we are: Eqs. 10.65 and 10.66 give us the fields, and the Lorentz force law determines the resulting force:

$$\begin{aligned} \mathbf{F} &= \frac{qQ}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\Big\{[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \\ &+ \frac{\mathbf{V}}{c}\times\Big[\hat{\boldsymbol{\mathscr{r}}}\times[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})]\Big]\Big\}, \end{aligned} \tag{10.67}$$

where $\mathbf{V}$ is the velocity of Q, and $\boldsymbol{\mathscr{r}}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{a}$ are all evaluated at the retarded time. The entire theory of classical electrodynamics is contained in that equation ... but you see why I preferred to start out with Coulomb's law.

Example 10.4

Calculate the electric and magnetic fields of a point charge moving with constant velocity.

Solution: Putting $\mathbf{a} = 0$ in Eq. 10.65,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{(c^2-v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\mathbf{u}.$$

In this case, using $\mathbf{w} = \mathbf{v}t$,

$$\mathscr{r}\mathbf{u} = c\boldsymbol{\mathscr{r}} - \mathscr{r}\mathbf{v} = c(\mathbf{r}-\mathbf{v}t_r) - c(t-t_r)\mathbf{v} = c(\mathbf{r}-\mathbf{v}t).$$

In Ex. 10.3 we found that

$$\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}.$$

In Prob. 10.14, you showed that this radical could be written as

$$Rc\sqrt{1-v^2\sin^2\theta/c^2},$$

where

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$$

is the vector from the *present* location of the particle to $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Thus

$$\boxed{\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}.} \tag{10.68}$$

Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an *extraordinary* coincidence, since the "message" came from the *retarded* position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is *reduced* by a factor $(1-v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1-v^2/c^2}$.

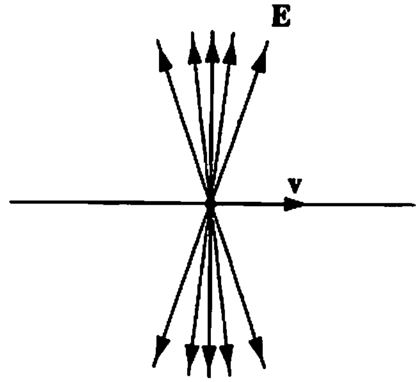

Figure 10.10

As for **B**, we have

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} = \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c},$$

and therefore

$$\boxed{\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).} \tag{10.69}$$

Lines of **B** *circle around* the charge, as shown in Fig. 10.11.

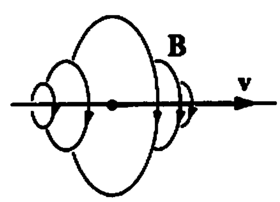

Figure 10.11

The fields of a point charge moving at constant velocity (Eqs. 10.68 and 10.69) were first obtained by Oliver Heaviside in 1888.[13] When $v^2 \ll c^2$ they reduce to

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}}; \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{q}{R^2}(\mathbf{v} \times \hat{\mathbf{R}}). \tag{10.70}$$

The first is essentially Coulomb's law, and the latter is the "Biot-Savart law for a point charge" I warned you about in Chapter 5 (Eq. 5.40).

[13]For history and references, see O. J. Jefimenko, *Am. J. Phys.* **62**, 79 (1994).