11.6 Mathematical Properties of the Space-Time of Special Relativity

In 3D, we talk about vectors and we say that there are transformations that leave the "norm" of the vectors invariant. Those are rotations, reflections, and translations. In 4D, the Lorentz transformations leave invariant also the norm of a 4-vector $(A_0, \vec{A})$.

We say that the "Lorentz group" is the set of transformations that leaves invariant

$$s^2(x, y) = (x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2$$

(x_0, x_1, x_2, x_3) $\quad$ (y_0, y_1, y_2, y_3)

already discussed in (11.1.5)

The transformations are
- ordinary rotations
- Lorentz transf. as in page 525
- reflections and translations in space-time

<u>Postulate</u>: the laws of physics must be "covariant" (that is, invariant in form) under transf. of the Lorentz group

Therefore, the laws of physics must involve Lorentz scalars, Lorentz vectors, Lorentz tensors, etc.

If we can show that the Max. Eqs. can be rewritten in terms of Lorentz scalar, vectors, or tensors, then they are covariant

In 3D, if we see a law of physics written in terms of vectors such as $\vec{F} = m\vec{a}$, then we know they are invariant under rotations, transl., and reflections. There is no need to "prove it". It is enough to see that there are vectors involved as opposed of say components. ($\vec{F} = m a_x$ is not invariant, for example).

Suppose there is a transformation

$$x'^{\alpha} = x'^{\alpha}(x^0, x^1, x^2, x^3) \; ; \quad \alpha = 0, 1, 2, 3$$

means "function of"

This is typically a linear transformation $x' = A x$.

4×4

<u>Scalars</u> do not change under the transformation.

<u>Vectors</u> transform as the coordinates do. For instance if

$$x' = ax + by \qquad \text{then} \qquad E_x' = a E_x + b E_y$$
$$y' = cx + dy \qquad\qquad\qquad E_y' = c E_x + d E_y$$

Note that "a" could be written as $\frac{\partial x'}{\partial x}$, "b" is $\frac{\partial x'}{\partial y}$, etc.

Thus, $E'_\alpha = \sum_\beta \frac{\partial x'_\alpha}{\partial x_\beta} E_\beta \; ; \; \alpha, \beta = x, y$

In general: $$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \qquad (11.61)$$

is the way in which a "vector" transforms when $K \to K'$.
Note that a repeated index is assumed summed over.

With regards to vectors, there are two kinds: "covariant" and "contravariant". Formally, we distinguish them by the location of the component index.

$$A^{\alpha} \longleftrightarrow \text{contravariant}$$

$$A_{\alpha} \longleftrightarrow \text{covariant}$$

The way they transform is

$$\boxed{A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}} \qquad \text{contravariant}$$

$$\boxed{B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}} \qquad \text{covariant}$$

Note the difference is that in one case we have $\frac{\partial x'^{\alpha}}{\partial x^{\beta}}$ with x' at the top, while the other one has it at the bottom.
In the 3D analog, in one case we use a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ while in the other it is the <u>inverse</u> of this matrix.

"Tensors of rank two" have 2 indices. If it is a "contravariant tensor of rank two", they transform as:

$$\boxed{F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}} \qquad (11.63)$$

A "covariant tensor of rank two" is

$$G'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} G_{\gamma\delta} \qquad (11.64)$$

Let us define a scalar product as:

$$\boxed{B \cdot A = B_{\alpha} A^{\alpha}} \qquad (11.66)$$

Consider $B' \cdot A'$ and see how it transforms:

$$B' \cdot A' = B'_{\alpha} A'^{\alpha} = \underbrace{\frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\bar{\beta}}}}_{\text{I am summing over } \alpha.} B_{\beta} A^{\bar{\beta}} = \underbrace{\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}}_{\delta^{\beta}{}_{\bar{\beta}}} B_{\beta} A^{\bar{\beta}} =$$

$$= B_{\beta} A^{\beta} = B \cdot A$$

Thus, (11.66) is invariant under the transf. $x' \to x$.

thus "scalar".

It can be shown that for our case of interest a vector covariant and one contravariant are related as:

$$\overset{\text{contravariant}}{A^\alpha} = (A^0, \vec{A}) \; ; \quad \overset{\text{covariant}}{A_\alpha} = (A^0, -\vec{A})$$

Thus: $B \cdot A = B_\alpha A^\alpha = B^0 A^0 - \vec{B} \cdot \vec{A}$ as we know is correct, from previous considerations.

Let us use the notation:

$$\text{contravariant} \rightarrow \partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla} \right) \qquad (11.76)$$

$$\text{covariant} \rightarrow \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

Proof in page 543

Then:

$$\boxed{\underbrace{\partial^\alpha A_\alpha}_{} = \partial_\alpha A^\alpha = \frac{\partial}{\partial x^0} A^0 + \vec{\nabla} \cdot \vec{A}} \qquad (11.77)$$

called a 4-divergence of a 4-vector A. It is a scalar.

$\partial_\alpha = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$ i.e. $+\vec{\nabla}$

$A^\alpha = (A^0, \vec{A})$ i.e. $+\vec{A}$

(11.77) is what appears as the Lorentz condition on the scalar and vector potentials (6.14).

As special case, the four-dimensional Laplacian operator is

$$\boxed{\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2} \qquad (11.78)$$

D'alembertian operator
(it is the "Laplace operator" of relativity)

As discussed in previous chapters, in the Lorenz gauge condition ~~are~~ the wave eqs. for $\vec{A}$ and Φ are:

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}$$

(see page 240)

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

with the condition

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla\cdot\vec{A} = 0 \qquad \text{(11.131) and (6.14)}$$

In view of (11.78), we can write

the cont. eq: $\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0$ becomes $\partial_\alpha J^\alpha = 0$

$$\Box \vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\Box \Phi = 4\pi\rho = \frac{4\pi}{c}(c\rho)$$

If we define $J^\alpha = (c\rho, \vec{J})$ and $A^\alpha = (\Phi, \vec{A})$ then we have:

$$\boxed{\Box A^\alpha = \frac{4\pi}{c} J^\alpha} \qquad (11.133)$$

and the gauge condition becomes

$$\boxed{\partial_\alpha A^\alpha = 0} \qquad \left[\text{since } \partial_\alpha = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) \quad (11.76)\right]$$

These are manifestly covariant just by the mere fact they are made of Lorentz scalars and vectors.

[Note that actually we have to prove that J^α is a 4-vector and for that purpose see discussion page 554 and 555.]

The discussion is based on "charge conservation".

Let us now write $\vec{E}$ and $\vec{B}$ in a covariant form. We know from (6.9) that

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\Phi$$

$$\vec{B} = \nabla \times \vec{A}$$

$\partial^1 = -\frac{\partial}{\partial x}$ (11.76)

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = \boxed{-\left(\partial^0 A^1 - \partial^1 A^0\right)}$$

$\partial^0 = \frac{\partial}{\partial x^0}$ (11.76) $\quad \boxed{\partial^\alpha = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)}$

$A^0 = \Phi$, $A^1 = A_x$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\partial^2 A^3 - \left(-\partial^3 A^2\right) = \boxed{-\left(\partial^2 A^3 - \partial^3 A^2\right)}$$

$A^2 = A_y$
$A^3 = A_z$
$\frac{\partial}{\partial y} = -\partial^2$
$\frac{\partial}{\partial z} = -\partial^3$

In general, define the antisymmetric field-strength tensor as:

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha}$$

$\Leftarrow$ it is a Lorentz tensor of rank 2 because it is the product of two 4-vectors i.e. transforms like (11.63) $\circledast$

$$\boxed{\begin{array}{cc} & \text{column} \\ \text{row} & \\ F^{\alpha\beta} = & \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \end{array}}$$

... this is F^{32}

Check:

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = -E_x \quad \checkmark$$

$$F^{10} = \partial^1 A^0 - \partial^0 A^1 = +E_x \quad \checkmark$$

$$F^{32} = \partial^3 A^2 - \partial^2 A^3 = B_x \quad \checkmark$$

etc, etc.

The "<u>dual" field-strength tensor</u> is defined as:

$$\boxed{\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_z \\ B_z & E_y & -E_z & 0 \end{pmatrix}}$$

It can also be shown to be a tensor of rank 2

Consider now the Maxwell Eqs. themselves:

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \hat{e}_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \cdots$$

Let us try with:

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

and see if it works.

$\underline{\underline{\beta=0}}$

$$\partial_\alpha F^{\alpha 0} = \frac{4\pi}{c} J^0$$

$$\partial_0 \underbrace{F^{00}}_{=0} + \underbrace{\partial_1}_{\frac{\partial}{\partial x^1_{=x}}} \underbrace{F^{10}}_{E_x} + \underbrace{\partial_2}_{\frac{\partial}{\partial x^2_{=y}}} \underbrace{F^{20}}_{E_y} + \underbrace{\partial_3}_{\frac{\partial}{\partial x^3_{=z}}} \underbrace{F^{30}}_{E_z} = \frac{4\pi}{c} J^0 = \frac{4\pi}{c} c\rho = 4\pi\rho$$

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 4\pi\rho$$

$\nabla \cdot \vec{E} = 4\pi\rho$ which is correct.

$\underline{\underline{\beta=1}}$

$$\underbrace{\partial_0}_{\frac{1}{c}\frac{\partial}{\partial t}} \underbrace{F^{01}}_{-E_x} + \underbrace{\partial_1}_{\frac{\partial}{\partial x}} \underbrace{F^{11}}_{=0} + \underbrace{\partial_2}_{\frac{\partial}{\partial y}} \underbrace{F^{21}}_{B_z} + \underbrace{\partial_3}_{\frac{\partial}{\partial z}} \underbrace{F^{31}}_{-B_y} = \frac{4\pi}{c} J^1$$

$$-\frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{4\pi}{c} J_x$$

which is the x component of $-\frac{\partial \vec{E}}{c\,\partial t} + \nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}$

The "other" Maxwell Equations, i.e. the "homogeneous ones" are:

$$\nabla\cdot\vec{B}=0, \quad \nabla\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$$

Try $\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta}=0}$

$\underline{\underline{\beta=0}}$

$$\partial_\alpha \mathcal{F}^{\alpha 0}=\partial_0\underbrace{\mathcal{F}^{00}}_{=0}+\partial_1\mathcal{F}^{10}+\partial_2\mathcal{F}^{20}+\partial_3\mathcal{F}^{30}=$$

$$=\underset{\frac{\partial}{\partial x}}{\partial_1}(+B_x)+\underset{\frac{\partial}{\partial y}}{\partial_2}(+B_y)+\underset{\frac{\partial}{\partial z}}{\partial_3}(+B_z)=\nabla\cdot\vec{B}=0 \quad \checkmark$$

$\leftarrow$ (11.76)

$\underline{\underline{\beta=1}}$

$$\partial_\alpha \mathcal{F}^{\alpha 1}=\underset{\frac{1}{c}\frac{\partial}{\partial t}}{\partial_0}\underbrace{\mathcal{F}^{01}}_{-B_x}+\partial_1\underbrace{\mathcal{F}^{11}}_{=0}+\underset{\frac{\partial}{\partial y}}{\partial_2}\underset{(-E_z)}{\mathcal{F}^{21}}+\underset{\frac{\partial}{\partial z}}{\partial_3}\underset{(+E_y)}{\mathcal{F}^{31}}=$$

$$=-\frac{1}{c}\frac{\partial B_x}{\partial t}-\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)=0 \Rightarrow \text{this is the } x \text{ component of } -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}-\nabla\times\vec{E}=0 \quad \checkmark$$

etc., etc.

So indeed the Maxwell equations are:

$$\boxed{\partial_\alpha F^{\alpha\beta}=\frac{4\pi}{c}J^\beta \quad \text{and} \quad \partial_\alpha \mathcal{F}^{\alpha\beta}=0}$$

By showing that the Maxwell Equations can be written in terms of two rank-two tensors $F^{\alpha\beta}$, $\mathcal{F}^{\alpha\beta}$ and two 4-vectors ∂_α, J^α, then we have shown that these equations are <u>covariant</u>.

Although we will <u>not</u> do it explicitly, the actual proof also needs that the Lorentz force equation

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

be shown to be covariant using $p^\alpha = (p_0, \vec{p}) = \left(\frac{E}{c}, \vec{p}\right)$ and $U^\alpha = (\gamma_u c, \gamma_u \vec{u})$ as shown in (11.36) with γ_u's being the usual $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ using u as velocity.

11.10 Transformation of Electromagnetic Fields

Since the components of $\vec{E}$ and $\vec{B}$ are part of $F^{\alpha\beta}$, and since $F^{\alpha\beta}$ is a tensor of rank 2, then we know it transforms as

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \qquad (11.146)$$

For a boost along the x axis, our study in section (11.16) can be used (although in that case the boost was along the z axis but it is trivial to change z into x). The transformation for the coordinates is:

$$x'_0 = \gamma(x_0 - \beta x_1)$$ now x_1 is "x", while in (11.16) was "z"

$$x'_1 = \gamma(x_1 - \beta x_0)$$

$$x'_2 = x_2$$

$$x'_3 = x_3$$

Consider $\alpha = 1,\ \beta = 0$

$$F'^{10} = \frac{\partial x'^{1}}{\partial x^{\gamma}} \frac{\partial x'^{0}}{\partial x^{\delta}} F^{\gamma\delta} = \frac{\partial x'^{1}}{\partial x_1} \frac{\partial x'^{0}}{\partial x_0} F^{10} +$$

$$+ \frac{\partial x'^{1}}{\partial x^{0}} \frac{\partial x'^{0}}{\partial x^{0}} F^{00} + \frac{\partial x'^{1}}{\partial x^{1}} \frac{\partial x'^{0}}{\partial x^{1}} F^{11} + \frac{\partial x'^{1}}{\partial x^{0}} \frac{\partial x'^{0}}{\partial x^{1}} F^{01} \quad (\text{only } \gamma,\delta = 0,1$$

But $F^{00} = F^{11} = 0$, and $F^{01} = -F^{10}$. Then:

$$\boxed{\underbrace{F'^{10}}_{E'_x}} = \Big(\underbrace{\frac{\partial x'^1}{\partial x^1}}_{\gamma}\underbrace{\frac{\partial x'^0}{\partial x^0}}_{\gamma} - \underbrace{\frac{\partial x'^1}{\partial x^0}}_{-\beta\gamma}\underbrace{\frac{\partial x'^0}{\partial x^1}}_{-\gamma\beta}\Big)\underbrace{F^{10}}_{E_x}$$

$$= (\gamma^2 - \beta^2\gamma^2)\, E_x = \underbrace{\gamma^2(1-\beta^2)}_{1}\, E_x = \boxed{E_x}$$

<u>Consider now $\alpha = 2$, $\beta = 0$</u>

$$\underbrace{F'^{20}}_{E'_y} = \frac{\partial x'^2}{\partial x^\gamma}\frac{\partial x'^0}{\partial x^\delta} F^{\gamma\delta} = \underbrace{\frac{\partial x'^2}{\partial x^2}}_{1}\frac{\partial x'^0}{\partial x^\delta} F^{2\delta} =$$

since x'^2 only depends on x_2, then $\gamma = 2$

It can only be 0 or 1, since x'^0 only depends on x^0 and x^1!

$$= \underbrace{\frac{\partial x'^0}{\partial x^0}}_{\gamma}\underbrace{F^{20}}_{E_y} + \underbrace{\frac{\partial x'^0}{\partial x^1}}_{-\beta\gamma}\underbrace{F^{21}}_{B_z}$$

$$\boxed{E'_y = \gamma(E_y - \beta B_z)}$$

Consider $\alpha = 2, \beta = 1$

$x'^2 = x^2$, thus $\gamma = 2$

$$\underbrace{F'^{21}}_{B_z'} = \frac{\partial x'^2}{\partial x^\gamma}\frac{\partial x'^1}{\partial x^\delta}F^{\gamma\delta} = \underbrace{\frac{\partial x'^2}{\partial x^2}}_{1}\frac{\partial x'^1}{\partial x^\delta}F^{2\delta} =$$

only 0 and 1

$$= \underbrace{\frac{\partial x'^1}{\partial x^0}}_{-\gamma\beta}\underbrace{F^{20}}_{E_y} + \underbrace{\frac{\partial x'^1}{\partial x^1}}_{\gamma}\underbrace{F^{21}}_{B_z} = \gamma(B_z - \beta E_y)$$

$$\boxed{B_z' = \gamma(B_z - \beta E_y)}$$

etc. etc. showing that (11.148) is correct.

These transformations show that $\vec{E}$ and $\vec{B}$ have <u>no</u> independent existence. A purely $\vec{E}$ field in one K may appear a mixture of $\vec{E}$ and $\vec{B}$ in another.

<u>Restriction</u>: a pure $\vec{E}$ <u>cannot</u> be transformed into a pure $\vec{B}$ and viceversa. This is obvious from (11.148) since for a pure $\vec{E}$ i.e. $B_1 = B_2 = B_3 = 0$ in K, still we get $E_1', E_2', E_3' \neq 0$

In more detail

$$E_1' = E_1 \qquad B_1' = 0$$
$$E_2' = \gamma E_2 \qquad B_2' = \gamma\beta E_3$$
$$E_3' = \gamma E_3 \qquad B_3' = -\gamma\beta E_2$$

In (11.148), the velocity v is in the direction x. Thus, $\vec{\beta} = \left(\frac{v}{c}, 0, 0\right)$

$$\boxed{\vec{\beta} \times \vec{E}'} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ v/c & 0 & 0 \\ E_1' & E_2' & E_3' \end{vmatrix} = -\hat{e}_y \underbrace{\frac{v}{c}}_{\beta} E_3' + \hat{e}_z \frac{v}{c} E_2' =$$

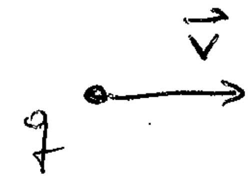
$$= -\hat{e}_y \underbrace{\beta \gamma E_3}_{+B_y'} + \hat{e}_z \underbrace{\beta \gamma E_2}_{-B_z'} = -\left(\hat{e}_y B_y' + \hat{e}_z B_z'\right) = \boxed{-\vec{B}'} \quad (11.150)$$

This is because of the sign of $\vec{v}$, different in our example than in the book.

to show (11.150)

Example

System K



System K' charge is <u>at rest</u>.

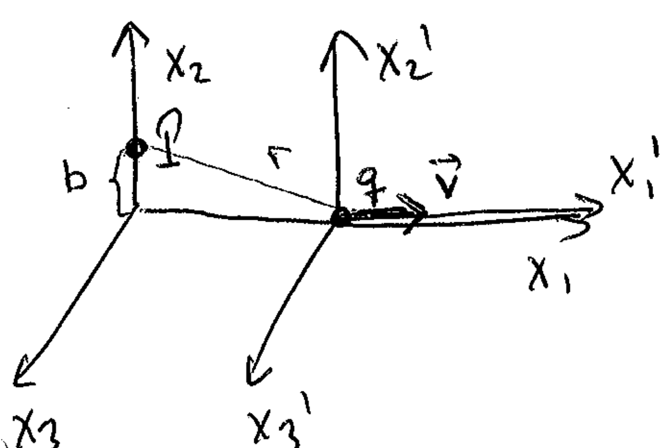
The fields transform as the inverse of (11.148)
(since in (11.148) we go from rest (k) to speed v(k'))

$$E_1 = E_1' \qquad B_1 = B_1'$$
$$E_2 = \gamma\left(E_2' + \beta B_3'\right) \qquad B_2 = \gamma\left(B_2' - \beta E_3'\right)$$
$$E_3 = \gamma\left(E_3' - \beta B_2'\right) \qquad B_3 = \gamma\left(B_3' + \beta E_2'\right)$$

$B' \leftrightarrow B$
$E' \leftrightarrow E$
and $\beta \to -\beta$

In system K', there is <u>no</u> magnetic field, thus

$$\boxed{B_1' = B_2' = B_3' = 0}$$



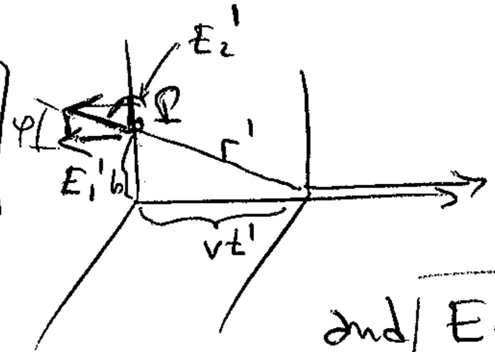
From the perspective of K',
the point P, where the observer is located, has coordinates:

$$x_3' = 0$$
$$x_2' = b$$
$$x_1' = -vt'$$

and $r' = \sqrt{b^2 + (vt')^2}$ is the distance P-charge in system K'.

$$\boxed{E_1' = -\frac{q}{r'^2} \cdot \underbrace{\frac{vt'}{r'}}_{\cos\varphi} = -\frac{q v t'}{r'^3}}$$

$$\boxed{E_2' = \frac{q}{r'^2} \underbrace{\frac{b}{r'}}_{\sin\varphi} = \frac{q b}{r'^3}}$$



and $\boxed{E_3' = 0}$ since K and K' move one with respect to the other along the x axis.

$E_1 = E_1'$

$E_2 = \gamma E_2'$

$E_3 = \gamma E_3' = 0$ ✓

$B_1 = 0$ ✓

$B_2 = -\gamma\beta E_3' = 0$ ✓

$B_3 = \gamma\beta E_2' \longrightarrow B_3 = \gamma\beta E_2' = \gamma\beta \frac{E_2}{\gamma} = \beta E_2$ ✓

$$\left.\begin{aligned} E_1 &= -\frac{q v t'}{r'^3} \\ E_2 &= \frac{\gamma q b}{r'^3} \end{aligned}\right\} \longrightarrow$$

Note that $t' = \gamma \left[t - \frac{v}{c^2} x_1\right]$

$x_0' = \gamma[x_0 - \beta x_1]$ (11.16)

$ct' = \gamma[ct - \beta x_1]$

$t' = \gamma\left[t - \frac{\beta}{c} x_1\right]$

Then:

$$E_1 = E_1' = \frac{-q v t'}{r'^3} = \frac{-q v \gamma \left[t - \cancel{\frac{\beta}{c} x_1}\right]}{\left[b^2 + v^2\gamma^2 t^2\right]^{3/2}}$$

Coordinate x_1 of P is $\underline{x_1 = 0}$ thus $t' = \gamma t$ at P.

$E_1 = E_1'$

$$\boxed{E_1 = \frac{-q v \gamma t}{(b^2 + v^2\gamma^2 t^2)^{3/2}}} \qquad (11.152)$$

$E_2 = \gamma E_2'$

while
$$\boxed{E_2 = \frac{\gamma q b}{(b^2 + v^2\gamma^2 t^2)^{3/2}}} \; ; \quad \boxed{E_3 = 0}$$

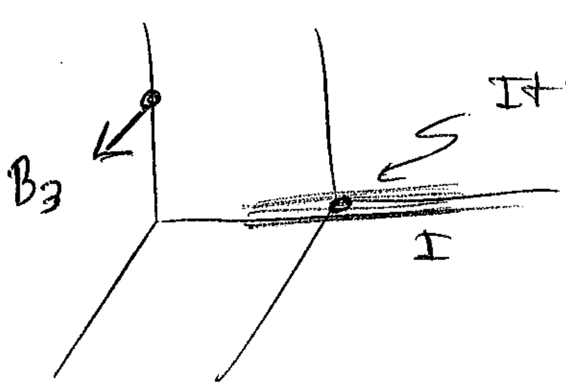
$B_3 = \beta E_2$

$$\boxed{B_1 = 0} \; ; \; \boxed{B_2 = 0} \; ; \; \boxed{B_3 = \frac{\gamma q \beta b}{(b^2 + v^2\gamma^2 t^2)^{3/2}}}$$

Note that as $v \to c$, then $\beta \to 1$ and E_2 and B_2 are almost the same.

as in a plane wave i.e. a relativistic particle produces a radiation field!

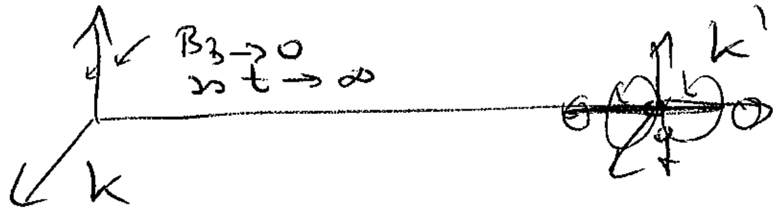
Even if $\gamma \approx 1$, i.e. $\beta \approx 0$, we still have a magnetic field that is as in the Ampere's law



It is like a current in a way since it is charge moving, thus we have

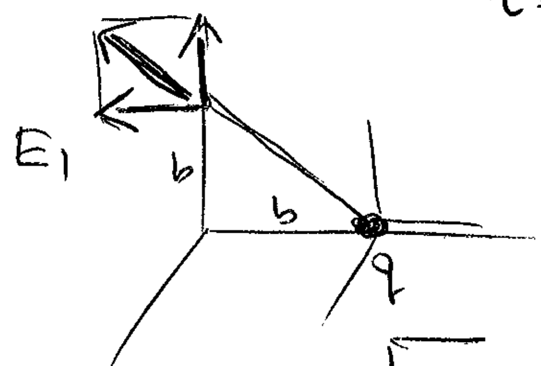
thus at P the $\vec{B}$ field is only along x_3

As $t \to \infty$, B_3 at $P \to 0$



Consider the time t such that $x_1 = b$ i.e.

$$t = \frac{b}{v}$$



In this case $\boxed{E_1 = \frac{-qv\gamma \frac{b}{v}}{\left(b^2 + v^2\gamma^2 \frac{b^2}{v^2}\right)^{3/2}}} = \frac{-q\gamma b}{b^3(1+\gamma^2)^{3/2}}$

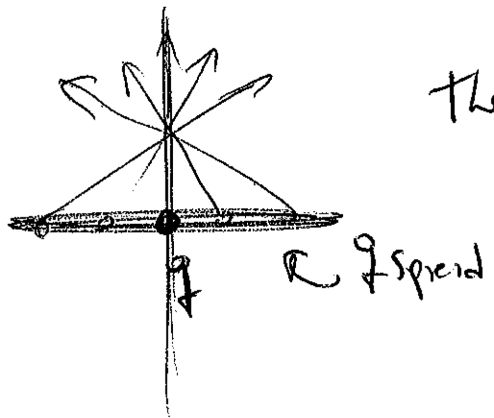
$$\boxed{= -\frac{q}{b^2} \cdot \frac{\gamma}{(1+\gamma^2)^{3/2}}}$$

while

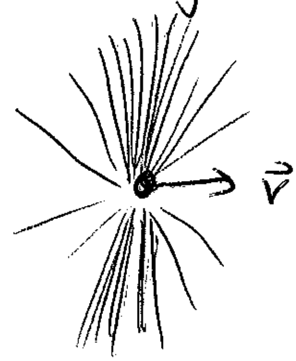
$$\boxed{E_2 = \frac{\gamma q b}{b^3(1+\gamma^2)^{3/2}} = \frac{q}{b^2} \cdot \frac{\gamma}{(1+\gamma^2)^{3/2}}}$$

It looks as the field created by a charge static at the present position, enhanced by $\frac{\gamma}{(1+\gamma^2)^{3/2}}$.

The charge is like "spread" along the direction of motion:



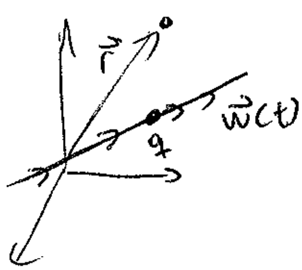
The net result is an electric field stronger ⊥ to the trajectory



(shown in the book in Eq. (11.154))

<u>NOTE</u>:

This is the same result we found before studying the fields produced by moving charges, via the retarded potentials!



In that calculation we never transform systems of coordinates, but did the calculation in a single system.

But we could have assumed the charge was <u>static</u> at a point, thus leading to an isotropic lines of force distribution, and then transform to the moving frame of the "observer" (going at $(-\vec{v})$) and find out what electric field they see.

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