2.6 Green Function for the Sphere

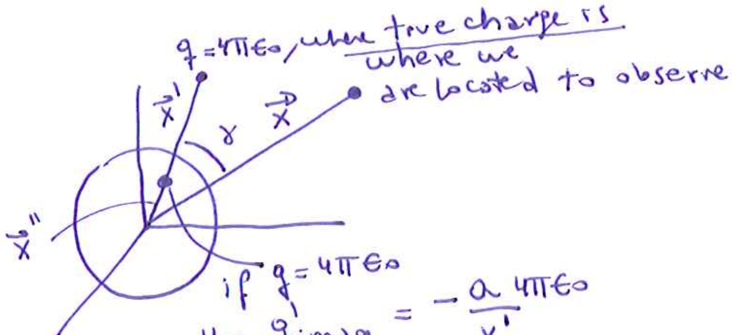
For a "unit source", the Green function is

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|}.$$

Then, for the problem of conducting sphere in the presence of a point charge, which we know is equivalent to the point charge plus its image, we have:

$$\boxed{G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} - \frac{a}{x'\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}\,'\right|}}$$

The reason is that "unit source" means $q = 4\pi\epsilon_0$



if $q = 4\pi\epsilon_0$

then $q_{image} = -\frac{a}{x'}4\pi\epsilon_0$

and $|\vec{x}\,''| = \frac{a^2}{x'}$, $\vec{x}\,'' = \frac{a^2}{x'}\vec{n} = \frac{a^2}{x'^2}\vec{x}\,'$

In spherical coordinates:

$$G(\vec{x},\vec{x}\,') = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{a}{x'\sqrt{x^2+\frac{a^4}{x'^4}x'^2-2x\frac{a^2}{x'^2}x'\cos\gamma}}$$

$$= \boxed{\frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{1}{\sqrt{\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma}}}.$$

It is clear that $G(\vec{x},\vec{x}\,') = G(\vec{x}\,',\vec{x})$ by doing $x \to x'$, $x' \to x$ in the last expression.

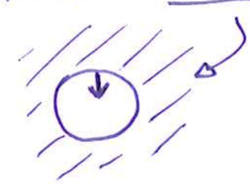
Also, if $x = a$ then

$$G(a\vec{n},\vec{x}\,') = \frac{1}{\sqrt{a^2+x'^2-2ax'\cos\gamma}} - \frac{1}{\sqrt{x'^2+a^2-2ax'\cos\gamma}} = 0$$

Then, $G = 0$ at the sphere as required in a Green function.

From Eq. (1.44) we know that to get the potential $\Phi(\vec{x})$ we not only need $G(\vec{x},\vec{x}\,')$ but also $\frac{\partial G}{\partial n'}$.

$\vec{n}\,'$ is the unit vector <u>normal inward ~~outward~~</u> from the sphere, because the volume of interest is <u>outside</u>



So $\frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial x'}$ and

$$-\frac{\partial G}{\partial x'}\bigg|_{x'=a} = -\Bigg[\left(-\frac{1}{2}\right)(\cancel{x}^2+x'^2-2\cancel{x}x'\cos\gamma)^{3/2}(2x'-2\cancel{x}\cos\gamma)$$
$$+\frac{1}{2}\left(\frac{x^2}{a^2}x'^2+a^2-2xx'\cos\gamma\right)^{3/2}\left(2\frac{x^2}{a^2}x'-2x\cos\gamma\right)\Bigg]\Bigg|_{x'=a} =$$

$$= -\Bigg[\left(-\frac{1}{2}\right)(a^2+x^2-2ax\cos\gamma)^{3/2}(2a-2x\cancel{\cos\gamma})$$
$$+\frac{1}{2}(a^2+x^2-2ax\cos\gamma)^{3/2}\left(2\frac{x^2}{a}-2x\cancel{\cos\gamma}\right)\Bigg]$$

$$= \left[\left(-\frac{1}{2}\right)2a + \frac{1}{2}\cdot 2\frac{x^2}{a}\right](a^2+x^2-2ax\cos\gamma)^{3/2}$$

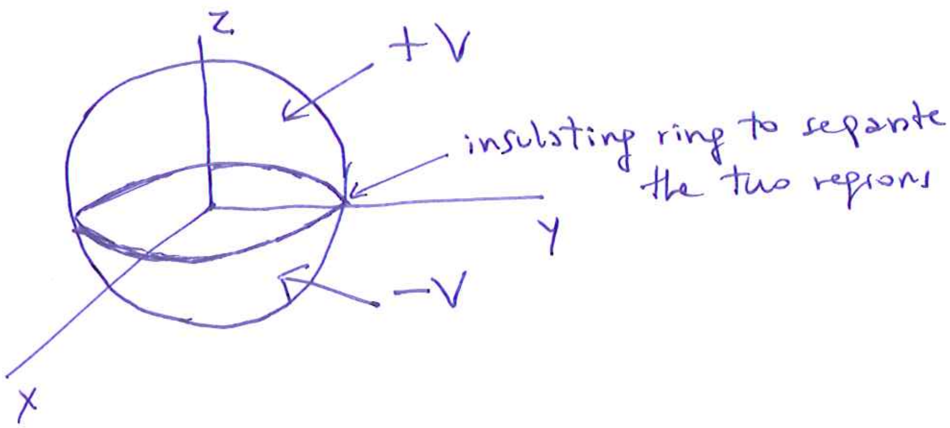
$$= -\frac{1}{a}\frac{(x^2-a^2)}{(a^2+x^2-2ax\cos\gamma)^{3/2}}$$

From Eq. (1.44), there is no $\rho(\vec{x}')$ thus the second term is everything:

$$\underbrace{\Phi(\vec{x})}_{\substack{\text{potential} \\ \text{outside} \\ \text{sphere}}} = -\frac{1}{4\pi} \oint_S \underbrace{\Phi(\vec{x}')}_{\substack{\text{potential at the} \\ \text{sphere, which} \\ \text{can be anything}}} \frac{\partial G}{\partial n'} da'$$

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \int \underbrace{d\Omega\, a^2}_{\text{"}da'\text{"}} \underbrace{\frac{-(x^2-a^2)}{a(a^2+x^2-2ax\cos\gamma)^{3/2}}}_{\frac{\partial G}{\partial n'}} \underbrace{\Phi(a,\theta',\phi')}_{\substack{\text{Potential at} \\ \text{sphere, whatever} \\ \text{it is.}}}$$

2.7 Conducting Sphere with Hemispheres at Different Potentials



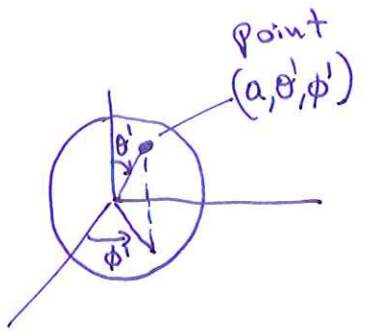
Eq. (2.19) gives:

actually this should be on the far right since γ depends on θ!

$$\Phi(x,\theta,\phi) = \frac{1}{4\pi}\int V \cdot \boxed{\frac{a(x^2-a^2)}{(a^2+x^2-2ax\cos\gamma)^{3/2}}} \cdot d\Omega'$$

from $(-)V$.

$$d\Omega' \to \int_0^{2\pi} d\phi' \Big(\underbrace{\int_0^{\pi/2} \sin\theta' d\theta'}_{\text{due to } +V} - \underbrace{\int_{\pi/2}^{\pi} \sin\theta' d\theta'}_{\text{due to } -V} \Big)$$



$$= \frac{Va(x^2-a^2)}{4\pi} \int_0^{2\pi} d\phi' \left[\int_0^1 d(\cos\theta') - \int_{-1}^0 d(\cos\theta') \right] \frac{1}{(a^2+x^2-2ax\cos\gamma)^{3/2}}$$

$$\int_0^{\pi/2} d\theta \sin\theta\, f(\cos\theta) = \int_1^0 (-du)\, f(u) = \int_0^1 du\, f(u)$$

$\cos\theta = u$

$-\sin\theta\, d\theta = du$

$\cos(\theta=0) = 1$

$\cos(\theta=\pi/2) = 0$

Now do $\theta' \to \pi - \theta''$ in the second integral

$$\cos\theta' \to \cos(\pi - \theta'') = -\cos\theta''$$

Then, $\int_{-1}^{0} d(\cos\theta') = -\int_{1}^{0} d(\cos\theta'') = \int_{0}^{1} d(\cos\theta'')$

and switch back $\theta'' \to \theta'$.

$$\boxed{\Phi(x,\theta,\phi) = \frac{Va(x^2-a^2)}{4\pi} \int_0^{2\pi} d\phi' \int_0^1 d(\cos\theta') \cdot \left[\frac{1}{(a^2+x^2-2ax\cos\gamma)^{3/2}} - \frac{1}{(a^2+x^2+2ax\cos\gamma)^{3/2}} \right]}$$

Note that

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

If we do $\theta' \to \pi - \theta'$

$$\cos\gamma = -\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

and for an overall switch of sign we do $\phi' \to \phi' + \pi$

so that $\cos(\phi-\phi') \to \cos(\phi-\phi'-\pi) = -\cos(\phi-\phi')$

and $\cos\gamma \to -\cos\gamma$

The framed result is the final result and the integrals can't be done unless we consider special cases

As special case consider the positive z axis.

Then, $\theta = 0$ and

$\cos\gamma = \cos\theta'$ since $\sin(\theta=0)=0$, $\cos(\theta=0)=1$

Thus, we get

$$\underset{\substack{\uparrow \\ x=z}}{\Phi(z,0,\phi) =} \frac{Va(x^2-a^2)}{4\pi}\int_0^{2\pi} d\phi' \underbrace{\int_0^1 d(\cos\theta')\left[\frac{1}{(a^2+x^2-2ax\cos\theta')^{3/2}} - \frac{1}{(a^2+x^2+2ax\cos\theta')^{3/2}}\right]}$$

$$\int_0^1 d\mu \left[\frac{1}{(a^2+x^2-2ax\mu)^{3/2}} - \frac{1}{(a^2+x^2+2ax\mu)^{3/2}}\right]$$

$$= \frac{1}{ax}(a^2+x^2-2ax\mu)^{-1/2} - \left(\frac{-1}{ax}\right)(a^2+x^2+2ax\mu)^{-1/2}\Bigg|_0^1$$

$$= \frac{1}{ax}\left[(a^2+x^2-2ax)^{-1/2} - (a^2+x^2)^{-1/2}\right]$$

$$+ \frac{1}{ax}\left[(a^2+x^2+2ax)^{-1/2} - (a^2+x^2)^{-1/2}\right]$$

$$= \frac{1}{ax}\left[\frac{1}{\sqrt{(a-x)^2}} - \frac{1}{\sqrt{a^2+x^2}} + \frac{1}{\sqrt{(a+x)^2}} - \frac{1}{\sqrt{a^2+x^2}}\right]$$

$$= \frac{1}{az}\left[\underbrace{\frac{1}{z-a} + \frac{1}{z+a}}_{\frac{z+a+z-a}{z^2-a^2}} - \frac{2}{\sqrt{a^2+z^2}}\right]$$

$$\Phi(z,0,\phi) = \frac{Va(z^2-a^2)}{4\pi} \cdot 2\pi \cdot \frac{1}{az}\left(\frac{2z}{z^2-a^2} - \frac{2}{\sqrt{a^2+z^2}}\right)$$

$$= \boxed{V\left[1 - \frac{(z^2-a^2)}{z\sqrt{z^2+a^2}}\right]} \qquad (2.22)$$

At $z=a$, $\Phi(a,0,\phi) = V$ as it has to be.

2.8 Orthogonal Functions and Expansions

The representation of solutions of problems via expansions in orthogonal functions is a powerful very general technique. The particular orthogonal set chosen depends on the symmetries of the problem.

Consider a variable ξ in the interval (a, b) and functions $U_n(\xi)$ $(n = 1, 2, \dots)$ such that they are square integrable and orthogonal i.e.

$$\int_a^b U_n^*(\xi)\, U_m(\xi)\, d\xi = 0 \quad \text{if } m \neq n$$
$$= 1 \quad \text{if } m = n$$

i.e. the functions are considered normalized to unity

$$\int_a^b |U_n(\xi)|^2\, d\xi = 1$$

Then, an arbitrary function $f(\xi)$ (with $\int_a^b |f(\xi)|^2 d\xi$ finite) can be expanded

$$f(\xi) = \sum_n a_n U_n(\xi) \quad \text{with} \quad a_n = \int_a^b U_n^*(\xi')\, f(\xi')\, d\xi'.$$

$$\left(\int_a^b U_m^* f(\xi)\, d\xi = \int_a^b U_m^* \sum_n a_n U_n\, d\xi = \sum_n a_n \delta_{nm} = a_m \right)$$

Then,
$$f(\xi) = \int_a^b \left[\sum_n U_m^*(\xi') U_n(\xi)\right] f(\xi')\, d\xi'$$

For this to be true then:

$$\boxed{\sum_{m=1}^{\infty} U_m^*(\xi') U_m(\xi) = \delta(\xi' - \xi)}$$

which is called the completeness or closure relation.

The most famous example are the sines and cosines used in Fourier series.

The orthonormal functions are

$$\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m x}{a}\right), \quad \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right)$$

where $m = 1, 2, \ldots$

and for $m=0$, the cosine function is $\frac{1}{\sqrt{a}}$

Then, for an arbitrary real function $f(x)$ in the interval $(-a/2, +a/2)$:

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right)\right]$$

with $$A_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi m x}{a}\right)dx$$

$$B_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\left(\frac{2\pi m x}{a}\right)dx$$

For 2 dimensions:

$$f(\xi,\eta) = \sum_n\sum_m a_{nm}\, U_n(\xi) V_m(\eta)$$

$$a_{nm} = \int_a^b d\xi \int_c^d d\eta \; U_n^*(\xi) V_m^*(\eta) f(\xi,\eta)$$

For (a,b) growing to infinity, then we use a continuum of functions. We can deduce it first taking a finite interval $(-a/2, a/2)$ and then sending to ∞ to arrive Fourier integral

$$U_m(x) = \frac{1}{\sqrt{a}} e^{i(2\pi m x/a)}$$

$$m = 0, \pm 1, \pm 2, \dots$$

$$\cancel{f(x)}\; A_m = \frac{1}{\sqrt{a}}\int_{-a/2}^{a/2} e^{-i(2\pi m x'/a)} f(x')\,dx'$$

$$f(x) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{+\infty} A_m\, e^{i\frac{2\pi m x}{a}}$$

If $a \to \infty$, it is convenient to make a change of variables $\frac{2\pi m}{a} = k$

Thus, $\sum_m \to \int_{-\infty}^{+\infty} dm = \frac{a}{2\pi} \int_{-\infty}^{+\infty} dk$

$$A_m \to \sqrt{\frac{2\pi}{a}}\, A(k)$$

Then:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k)\, e^{ikx}\, dk$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x)\, dx$$

and the orthogonality condition is now

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k-k')x}\, dx = \delta(k-k')$$

Dirac delta

and the completeness becomes

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-x')}\, dk = \delta(x-x')$$

2.9 Separation of Variables

This is a method to solve differential eqs.

Consider the Laplace equation in rectangular coordinates:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

<u>Assume</u> $\phi(x,y,z) = X(x)\, Y(y)\, Z(z)$

Substitute, and divide by ϕ:

$$\underbrace{\frac{1}{X(x)} \frac{d^2 X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y(y)} \frac{d^2 Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z(z)} \frac{d^2 Z}{dz^2}}_{\gamma^2 = \alpha^2 + \beta^2} = 0$$

Considering $-\alpha^2 \quad -\beta^2 \quad \gamma^2 = \alpha^2+\beta^2$

then we have an infinite number of solutions

since the first eq. $\frac{1}{X} \frac{d^2 X}{dx^2} = -\alpha^2$

has as solution $e^{\pm i\alpha x} = X(x)$ $\left(\frac{d^2 X}{dx^2} = (\pm i\alpha)^2 e^{\pm i\alpha x} = -\alpha^2 e^{\pm i\alpha x} = -\alpha^2 X \right)$

The second leads to $Y(y) = e^{\pm i\beta y}$

and the third:

$$Z(z) = e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

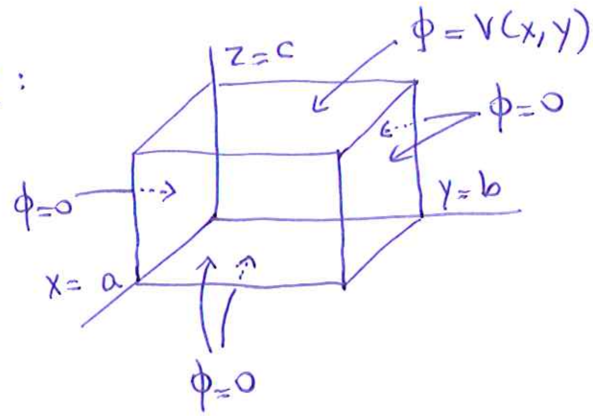
no "i"

Thus:

$$\Phi = e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

α, β are arbitrary and they can be found via the <u>boundary conditions</u>.

<u>Example</u>:



5 faces have $\phi = 0$
1 face has $\phi = V(x,y)$

Find potential anywhere <u>inside</u> the box.

Since $\phi = 0$ for $x = 0$, then $X(x) = \sin \alpha x$ ($\cos \alpha x$ would not vanish for any α)

$\phi = 0$ for $y = 0$, then $Y(y) = \sin \beta y$

$\phi = 0$ for $z = 0$, then $Z(z) = \sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right)$

mixture of $e^{+\sqrt{\alpha^2+\beta^2}\, z}$ and $e^{-\sqrt{\alpha^2+\beta^2}\, z}$

From $\phi = 0$ at $x = a$ we deduce

$$\sin(\alpha a) = 0 \quad \text{or} \quad \alpha_n = \frac{n\pi}{a}$$

From $\phi = 0$ at $\cancel{x}\, y = b$ then $\sin(\beta a) = 0 \rightarrow \beta_m = \frac{m\pi}{b}$

Finally ~~from~~ defining $\gamma_{nm} = \sqrt{\alpha_n^2 + \beta_n^2} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$,

we get:

$$\boxed{\phi_{nm} = \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)}$$

$$\boxed{\phi(x,y,z) = \sum_{n,m=1}^{\infty} \underbrace{A_{nm}}_{\text{to be determined}} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)}$$

We still need to satisfy $\phi = V(x,y)$ at $z = c$ i.e.

$$\boxed{V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} c)}$$

We know $\frac{2}{a}\int_{-a/2}^{a/2} \sin\left(\frac{2\pi m x}{a}\right)\sin\left(\frac{2\pi n x}{a}\right) dx = \delta_{nm}$

from page 68 book.

If I do $2x = u$ we get

$$\frac{2}{a}\int_{-a}^{a} \sin\left(\frac{\pi m u}{a}\right)\sin\left(\frac{\pi n u}{a}\right)\frac{du}{2} = \delta_{nm}$$

$$\frac{2}{a}\Big(\int_0^a + \underbrace{\int_{-a}^0}\Big) = \delta_{nm} \quad ; \quad \frac{4}{a}\int_0^a \cdots = \delta_{nm}$$

$\hookrightarrow u = -v$

$$\int_{-a}^0 \to \int_a^0 (-dv)$$

and u back to x

$$\frac{2}{a}\int_0^a \sin\left(\frac{\pi m \cancel{x}}{a}\right) \sin\left(\frac{\pi n \cancel{x}}{a}\right) d\cancel{x} = \delta_{nm} \qquad (1)$$

Same in y direction

$$\frac{2}{b}\int_0^b \sin\left(\frac{\pi m y}{b}\right) \sin\left(\frac{\pi n y}{b}\right) dy = \delta_{nm} \qquad (2)$$

Then:

$$\int_0^a dx \int_0^b dy \; V(x,y) \sin(\alpha_l x) \sin(\beta_p y) =$$

$$= \int_0^a dx \int_0^b dy \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \cdot$$

$$\cdot \sin(\alpha_l x) \sin(\beta_p y)$$

orthogonality (1) and (2)

$$= \frac{a}{2}\frac{b}{2} \sinh(\gamma_{lp} c) A_{lp}$$

Thus, $$\boxed{A_{lp} = \frac{4}{ab\sinh(\gamma_{lp} c)} \int_0^a dx \int_0^b dy \; V(x,y) \sin(\alpha_l x) \sin(\beta_p y)}$$

$l, p =$ integers