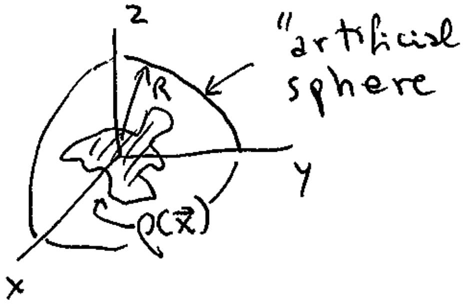
4.1 Multipole Expansion

Let us assume a localized distribution of charge, meaning the charge is confined to a finite region of space. Basically it means that it all fits inside a sphere of radius R.



Outside the sphere we should not have terms in the potential involving r^ℓ because $\xrightarrow[r\to\infty]{} \infty$ and there is no reason for that.

Thus, from the general expression (3.61) we keep

$$\Phi(r,\theta,\phi) \underset{r>R}{=} \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \underbrace{B_{\ell m}} r^{-(\ell+1)} Y_{\ell m}(\theta,\phi)$$

↳ The author says that for later convenience

$$B_{\ell m} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{2\ell+1} q_{\ell m}$$

i.e. it is simply "rewritten"

$$\boxed{\Phi(r>R,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1} q_{\ell m}\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}}$$

This expression is called a "multipole expansion".

$l=0 \rightarrow$ monopole term

$l=1 \rightarrow$ dipole terms

etc.

We need to find "q_{lm}". We know from previous considerations that:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

To link one with the other, we will use (3.70)

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\,\frac{r_<^l}{r_>^{l+1}}\, Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

where $r_<$ is the smaller of $|\vec{x}|$ and $|\vec{x}'|$
$r_>$ is the larger of $|\vec{x}|$ and $|\vec{x}'|$

Then:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\, d^3x'\; 4\pi \sum_{l,m} \frac{1}{2l+1}\,\frac{r_<^l}{r_>^{l+1}}\, Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

outside R

$r_< = r'$

$r_> = r$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l,m}\frac{4\pi}{2l+1}\int d^3x'\,\rho(\vec{x}')\,\frac{r'^l}{r^{l+1}}\, Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

Then, by comparison:

$$\boxed{q_{\ell m} = \int Y_{\ell m}^{*}(\theta',\phi')\, r'^{\ell}\, \rho(\vec{x}')\, d^3x'}$$

These are the "multipole moments".

Consider the first few:

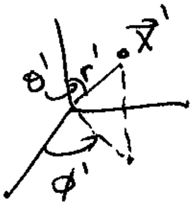
total charge

$$q_{00} = \int \underbrace{Y_{00}^{*}(\theta',\phi')}_{1/\sqrt{4\pi}} (r')^{0}\, \rho(\vec{x}')\, d^3x' = \frac{1}{\sqrt{4\pi}}\, q$$

$$q_{11} = \int \underbrace{Y_{11}^{*}(\theta',\phi')}_{-\sqrt{\frac{3}{8\pi}}\sin\theta' e^{-i\phi'} \text{ from page 109}} r'\, \rho(\vec{x}')\, d^3x' =$$

due to *

$$= -\sqrt{\frac{3}{8\pi}} \int \underbrace{r'\sin\theta'(\cos\phi' - i\sin\phi')}\, \rho(\vec{x}')\, d^3x' =$$



Then $x' = r'\sin\theta'\cos\phi'$
$y' = r'\sin\theta'\sin\phi'$

$$= -\sqrt{\frac{3}{8\pi}} \int (x' - i y')\, \rho(\vec{x}')\, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}}\, (p_x - i p_y)$$

$$q_{10} = \int Y_{10}^*(\theta',\phi')\, r'\, \rho(\vec{x}')\, d^3x'$$

$$= \sqrt{\frac{3}{4\pi}} \int \underbrace{\cos\theta\, r'}_{z'}\, \rho(\vec{x}')\, d^3x'$$

$$= \sqrt{\frac{3}{4\pi}}\, p_z$$

$$q_{22} = \int Y_{22}^*(\theta',\phi')\, r'^2\, \rho(\vec{x}')\, d^3x'$$

$$\overset{\text{page } 109}{=} \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int \underbrace{\sin^2\theta'\, e^{-2i\phi'}\, r'^2}_{\substack{r'^2 \sin^2\theta' (\cos\phi' - i\sin\phi')^2 \\ = (x' - iy')^2}}\, \rho(\vec{x}')\, d^3x'$$

$$= \frac{1}{12}\sqrt{\frac{15}{2\pi}} \underbrace{\int d^3x'\, \rho(\vec{x}')\; 3\left(x'^2 - 2i x'y' + \overset{(-1)}{i^2} y'^2\right) \cancel{\rho(\vec{x}')}}$$

Introduce:

$$Q_{11} \overset{4.9}{=} \int (3x'^2 - r'^2)\, \rho(\vec{x}')\, d^3x'$$

$$Q_{12} = \int 3x'y'\, \rho(\vec{x}')\, d^3x'$$

$$Q_{22} = \int (3y'^2 - r'^2)\, \rho(\vec{x}')\, d^3x'$$

$$Q_{11} - 2i\,Q_{12} - Q_{22} =$$

$$= \int \rho(\vec{x}')\,d^3x' \left[\underbrace{3x'^2 - \cancel{r'^2} - 2i\,3x'y' - (3y'^2 - \cancel{r'^2})}_{\substack{3x'^2 - 3y'^2 - 3.2i\,x'y' = \\ = 3(x'^2 - 2i x'y' - y'^2)}} \right]$$

which is what we had.

Then $$\boxed{q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\,(Q_{11} - 2i\,Q_{22} - Q_{22})}$$

Consider now q_{21}:

$$q_{21} = \int \underbrace{Y^{*}_{21}(\theta',\phi')}_{\substack{\text{page 109} \\ -\sqrt{\frac{15}{8\pi}}\sin\theta'\cos\theta' e^{-i\phi'}}} r'^2 \rho(\vec{x}')\,d^3x' =$$

$$= \int \rho(\vec{x}')\,d^3x' \left(-\sqrt{\frac{15}{8\pi}}\right) \underbrace{r'^2 \sin\theta' \cos\theta' (\cos\phi' - i\sin\theta')}_{\underbrace{r'\cos\theta'(r'\sin\theta'\cos\phi' - i r'\sin\theta'\sin\theta')}_{z'(x'-iy')}}$$

$$= -\frac{1}{3}\sqrt{\frac{15}{8\pi}} \int 3z'(x'-iy')\,\rho(\vec{x}')\,d^3x'$$

left as exercise

$$= \boxed{-\frac{1}{3}\sqrt{\frac{15}{8\pi}}\,(Q_{13} - i\,Q_{23}) = q_{21}}$$

etc.

We have used the definitions:

"electric dipole moment" $$\vec{P} = \int \vec{x}' \rho(\vec{x}')\, d^3x'$$

"quadrupole moment tensor" $$Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij})\, \rho(\vec{x}')\, d^3x'$$

Then:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} =$$

$$= \frac{1}{4\pi\epsilon_0}\Big[\underbrace{4\pi\, q_{00} \frac{Y_{00}}{r}}_{l=m=0} + \underbrace{\frac{4\pi}{3} q_{11} \frac{Y_{11}(\theta,\phi)}{r^2}}_{l=1,\, m=1} + \underbrace{\frac{4\pi}{3} q_{10} \frac{Y_{10}(\theta,\phi)}{r^2}}_{l=1,\, m=0} +$$

$$+ \underbrace{\frac{4\pi}{3} q_{1-1} \frac{Y_{1-1}(\theta,\phi)}{r^2}}_{l=1,\, m=-1} + \cdots \Big]$$

$$4\pi q_{00} \frac{Y_{00}}{r} = 4\pi \frac{1}{\sqrt{4\pi}} q \cdot \frac{1}{\sqrt{4\pi}} \frac{1}{r} = \boxed{\frac{q}{r}}$$

$$\frac{4\pi}{3} q_{11} \frac{Y_{11}}{r^2} = \frac{4\pi}{3}\left(-\sqrt{\frac{3}{8\pi}}\right)\left(\frac{p_x - ip_y}{r^2}\right)\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta\, e^{i\phi} =$$

$$= \frac{1}{2} \frac{(p_x - ip_y)}{r^3} \underbrace{r\sin\theta(\cos\phi + i\sin\theta)}_{(x+iy)}$$

$$= \boxed{\frac{1}{2} \frac{(p_x - ip_y)(x+iy)}{r^3}}$$

$$\frac{4\pi}{3} q_{10} \frac{Y_{10}}{r^2} = \frac{4\pi}{3} \sqrt{\frac{3}{4\pi}}\, p_z \sqrt{\frac{3}{4\pi}} \frac{\cos\theta}{r^2} = \boxed{\frac{p_z}{r^3} z}$$

$$\frac{4\pi}{3} q_{1-1} \frac{Y_{1-1}}{r^2} \underset{\uparrow}{=} \underset{\uparrow}{\frac{4\pi}{3}}(-1)\left(-\sqrt{\frac{3}{8\pi}}\right)\frac{(p_x + ip_y)}{r^2}(-1)\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta\, e^{-i\phi} =$$

(4.7)

$q_{1-1} = (-1)\, q_{11}^*$

(3.54)

$Y_{1,-1} = (-1)\, Y_{11}^*$

$$= \frac{1}{2} \frac{(p_x + ip_y)}{r^3} \underbrace{r\sin\theta(\cos\phi - i\sin\phi)}_{(x-iy)}$$

$$= \frac{1}{2} \frac{(p_x + ip_y)(x-iy)}{r^3}$$

Adding the three terms of $l=1$ gives

$$\frac{1}{r^3}\left[\frac{1}{2}(p_x - i p_y)(x+iy) + p_z z + \frac{1}{2}(p_x + i p_y)(x-iy)\right]$$

$$= \frac{1}{r^3}\Big[\frac{1}{2}(p_x x + i \cancel{p_x} y - i \cancel{p_y} x + p_y y) + p_z z$$
$$+ \frac{1}{2}(p_x x - i \cancel{p_x} y + i \cancel{p_y} x + p_y y) \Big] =$$

$$= \boxed{\frac{1}{r^3}\,\vec{p}\cdot\vec{x}}$$ which is correct, see (4.10)

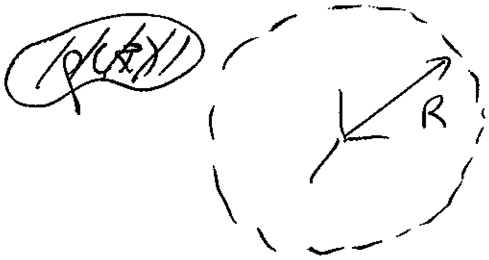
In general:

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j=1,2,3} Q_{ij}\,\frac{x_i x_j}{r^5} + \cdots\right]} \quad (4.10)$$

If we want $\vec{E}$, we use $\vec{E} = -\nabla\Phi$ in spherical coordinates see back of book.

Consider again a localized charge distribution $\rho(\vec{x})$. We want to calculate the integral of $\vec{E}$ over the entire volume of a sphere of radius R, first outside $\rho(\vec{x})$ i.e with charge inside

and then as



Let us start in general. The origin of coordinates will be at the center of the sphere. Let us calculate the integrated $\vec{E}$:

$$\int_{r<R} \vec{E}(\vec{x})\, d^3x \underset{\substack{\uparrow \\ \vec{E}=-\nabla\phi}}{=} -\int_{r<R} \nabla\Phi\, d^3x = -\int_{r=R} R^2 d\Omega\, \Phi\, \vec{n} =$$

$$\underset{\substack{\uparrow \\ \Phi=\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}}}{=} \frac{-R^2}{4\pi\epsilon_0}\int d^3x'\, \rho(\vec{x}') \underbrace{\int d\Omega \frac{\vec{n}}{|\vec{x}-\vec{x}'|}}$$

↳ let us now calculate this part.

$$\int_{r=R} d\Omega \quad \frac{\vec{n}}{|\vec{x}-\vec{x}'|} = \uparrow$$

$\vec{n} = \vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta$

From page 108

$$\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{10}$$

$$\sin\theta\cos\phi = -\sqrt{\frac{8\pi}{3}}\, \mathrm{Re}\, Y_{11}$$

$$= -\sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}+Y_{11}^*}{2}\right) = -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\underbrace{\left[Y_{11} + Y_{1-1}(-1)\right]}_{(-Y_{1-1}^* + Y_{11}^*)}$$

$$\sin\theta\sin\phi = -\sqrt{\frac{8\pi}{3}}\, \mathrm{Im}\, Y_{11}$$

$$= -\sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}-Y_{11}^*}{2i}\right) = -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\underbrace{\left(\frac{Y_{11} - Y_{1-1}(-1)}{i}\right)}_{\left(\frac{-Y_{1-1}^* - Y_{11}^*}{i}\right)}$$

Thus, $\vec{n}$ has only components involving $l = 1$.

$$= \int_{r=R} d\Omega \quad \vec{n} \quad 4\pi \sum_{m=-1}^{1} \underset{\substack{\uparrow \text{From} \\ 2l+1}}{\frac{1}{3}} \frac{r_<}{r_>^2} \, Y_{1m}^*(\theta',\phi')\, Y_{1m}(\theta,\phi) =$$

$$= \frac{4\pi}{3}\frac{r_<}{r_>^2}\sum_{m=-1}^{+1} Y_{1m}^*(\theta',\phi') \int_{r=R} d\Omega \quad \vec{n}\, Y_{1m}(\theta,\phi)$$

$$\uparrow \quad \vec{i}\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\overbrace{\left(Y_{11} - Y_{1-1}\right)}^{-Y_{1-1}^* + Y_{11}^*}$$

$$+ \vec{j}\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\left(\frac{Y_{11}+Y_{1-1}}{i}\right)$$

$$+ \vec{k}\sqrt{\frac{4\pi}{3}}\, Y_{10}$$

$$\int d\Omega\ \vec{k}\sqrt{\frac{4\pi}{3}}\,Y_{10}\cdot Y_{1m}(\theta,\phi) \underset{\uparrow}{=} \delta_{m0}\ \vec{k}\sqrt{\frac{4\pi}{3}}$$

$$(3.55)\qquad Y_{10}=Y_{10}^{*}$$

$$\int d\Omega\ \vec{i}\left(\frac{-1}{2}\sqrt{\frac{8\pi}{3}}\right)\left(-Y_{1-1}^{*}+Y_{11}^{*}\right)Y_{1m} = \vec{i}\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\left(-\delta_{m,-1}+\delta_{m,1}\right)$$

$$\int d\Omega\ \vec{j}\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\frac{\left(-Y_{1-1}^{*}-Y_{11}^{*}\right)}{i}\,Y_{1m} = \vec{j}\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)(+i)\left(\delta_{m,-1}+\delta_{m,1}\right)$$

Then: $$\frac{4\pi}{3}\frac{r_<}{r_>^2}\Bigg[\sqrt{\frac{4\pi}{3}}\,\vec{k}\ \overbrace{Y_{10}^{*}(\theta',\phi')}^{Y_{10}} + \left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\vec{i}\left(-Y_{1-1}^{*}+Y_{10}^{*}\right) + \left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)(+i)\,\vec{j}\left(Y_{1-1}^{*}+Y_{11}^{*}\right)\Bigg]$$

$$= \frac{4\pi}{3}\frac{r_<}{r_>^2}\left[\vec{k}\cos\theta' + \vec{i}\sin\theta'\cos\phi' + \vec{j}\sin\theta'\sin\phi'\right]$$

$$= \frac{4\pi}{3}\frac{r_<}{r_>^2}\,\vec{n}'.$$ Thus:

$$\boxed{\begin{aligned}\int_V \vec{E}(\vec{x})\,d^3x &= -\frac{R^2}{4\pi\epsilon_0}\int d^3x'\,\rho(\vec{x}')\underbrace{\int_{r=R} d\Omega\,\frac{\vec{n}}{|\vec{x}-\vec{x}'|}}_{\frac{4\pi}{3}\frac{r_<}{r_>^2}\vec{n}'}\\ &= \frac{-R^2}{3\epsilon_0}\boxed{\frac{r_<}{r_>^2}}\xrightarrow[\text{sorry}]{}\int d^3x'\,\vec{n}'\,\rho(\vec{x}')\end{aligned}}$$

Now consider sphere R entirely encloses $\rho(\vec{x}')$. Then:

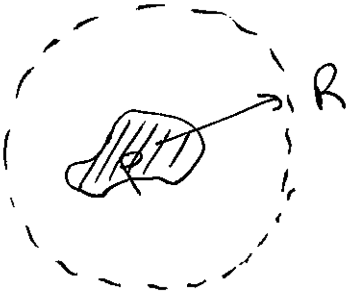
$$r_< = r' \,, \quad r_> = R \quad \text{and}$$

$$\int_V \vec{E}(\vec{x})\, d^3x = -\frac{R^2}{3\epsilon_0}\int \frac{r'}{R^2}\, d^3x'\, \vec{n}'\rho(\vec{x}')$$

$$= \underbrace{-\frac{1}{3\epsilon_0}\int d^3x'\, \rho(\vec{x}')\, \underbrace{r'\vec{n}'}_{\vec{x}'}}_{\text{Eq.(4.8)}} = -\frac{\vec{P}}{3\epsilon_0}$$

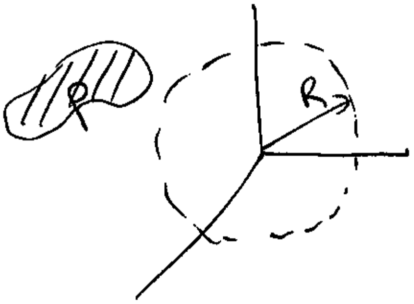
In summary:

$$\boxed{\int_V \vec{E}(\vec{x})\, d^3x = \frac{-\vec{P}}{3\epsilon_0}} \quad \text{for}$$



Note that result is <u>independent of R</u>.

Now consider that the charge is all <u>outside of</u> R



Then $r_< = R$
$r_> = r'$

$$\int_V \vec{E}(\vec{x})\, d^3x = -\frac{R^2}{3\epsilon_0}\int \frac{R}{r'^2}\, d^3x'\, \vec{n}'\rho(\vec{x}')$$

$$= -\frac{R^3}{3\epsilon_0}\underbrace{\int d^3x'\, \frac{\vec{n}'}{r'^2}\,\rho(\vec{x}')}_{} =$$

(1.5) says that

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\, d^3x'$$

Consider $\vec{x}=\vec{0}$ (origin) in this formula.

Then $\vec{E}(\vec{0}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{(-\vec{x}')\,d^3x'}{|\vec{x}'|^3}$

$$= -\frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{\cancel{r'}\vec{n}'}{r'^{\cancel{3}\,2}}\, d^3x'$$

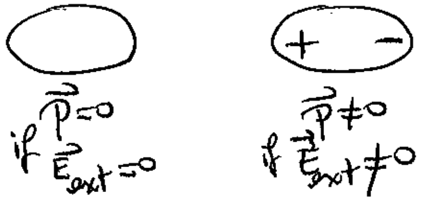
$$= \frac{-R^3}{3\epsilon_0}\left(-4\pi\epsilon_0\,\vec{E}(\vec{0})\right)$$

$$= \boxed{\frac{4\pi}{3} R^3\,\vec{E}(\vec{0})}$$

In other words, the average value of $\vec{E}$ over a sphere outside the charge is the value at the center of the sphere.

4.3 Electrostatics with Ponderable Media

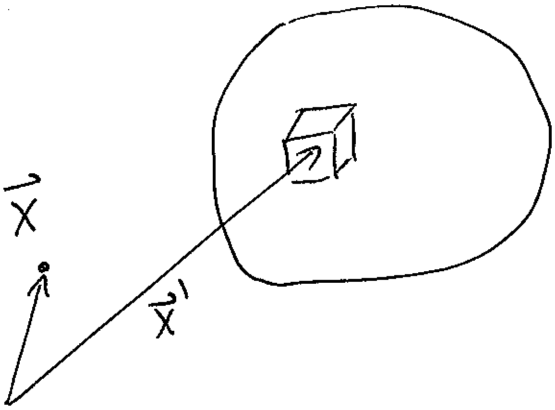
Many materials will react to an electric field by creating microscopic dipoles. For example a material made out of molecules will polarize its distribution of charge



The medium induces an "electric polarization" $\vec{P}$

$$\vec{P}(\vec{x}) = \underset{\substack{\text{molecules}\\ \text{or atoms}\\ \text{or unit cells}}}{\sum_i} \underset{\substack{\#\text{ of atoms}\\ \text{per unit cell}}}{N_i} \langle \vec{p}_i \rangle$$

From a <u>macroscopic</u> point of view, we use linear superposition and we build up the full potential.



$$\underbrace{\Delta\Phi(\vec{x},\vec{x}')}_{\substack{\text{potential felt}\\ \text{at } \vec{x} \text{ due}\\ \text{to small cube}\\ \text{centered at } \vec{x}'}} = \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(\vec{x}')\,\Delta v}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')\,\Delta}{|\vec{x}-\vec{x}'|^3} + \cdots\right]$$

From 4.10

Let us assume there are no quadrupole contributions so "...." is dropped, and now let us integrate over all the small cubes:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right] \underset{\substack{\uparrow\\ \text{See page 29}}}{=}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \underbrace{\vec{P}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}_{\text{integrating "by parts"}}\right] =$$

$$\underbrace{\int \nabla'\cdot(\phi\vec{F})}_{\substack{=0 \text{ if}\\ \text{all is}\\ \text{bounded}}} = \underbrace{\int(\nabla'\phi)\cdot\vec{F} + \int\phi(\nabla'\cdot\vec{F})}_{\nearrow\ \int\vec{F}\cdot(\nabla'\phi) = -\int\phi(\nabla'\cdot\vec{F})}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\underbrace{\left[\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right]}_{\text{like "effective charge density"}}$$

Thus, $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$ becomes

$$\underbrace{\underbrace{\nabla^2\phi}_{\nabla\cdot\nabla\phi}}_{-\nabla\cdot\vec{E}} = -\frac{\rho_{eff}}{\epsilon_0} = -\frac{1}{\epsilon_0}(\rho - \nabla\cdot\vec{P})$$

$$\boxed{\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho - \nabla\cdot\vec{P})}$$

Defining $\vec{D} \overset{\text{def}}{=} \epsilon_0 \vec{E} + \vec{P}$

then

$$\nabla\cdot\vec{D} = \nabla\cdot(\epsilon_0\vec{E}+\vec{P})$$

$$= \epsilon_0 \underbrace{\nabla\cdot\vec{E}}_{\frac{1}{\epsilon_0}(\rho-\nabla\cdot\vec{P})} + \nabla\cdot\vec{P} = \rho$$

$$\boxed{\nabla\cdot\vec{D}=\rho}$$

Before a solution of this type of problems is reached we need extra information. We need a constitutive relation connecting $\vec{D}$ and $\vec{E}$. Namely we need information about the properties of the material. It will be assumed that the relation is linear, excluding ferroelectric materials. We will also assume the medium is isotropic:

$$\boxed{\vec{P} = \epsilon_0\, \underline{\underline{\chi_e}}\, \vec{E}}$$

electric susceptibility of the medium

$$\vec{D} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E}$$
$$= \underbrace{\epsilon_0(1+\chi_e)}_{\epsilon}\vec{E}$$

$$\boxed{\epsilon = \epsilon_0(1+\chi_e)} \; ; \; \underline{\underline{\frac{\epsilon}{\epsilon_0}}} = 1+\chi_e$$

electric permittivity

dielectric constant

If χ_e is uniform in space, then

$$\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho - \nabla\cdot\vec{P}) = \frac{1}{\epsilon_0}(\rho - \epsilon_0\chi_e\nabla\cdot\vec{E})$$

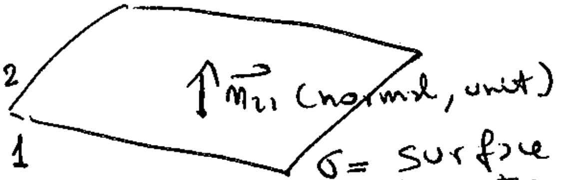
$$\nabla\cdot\vec{E} = \frac{\rho/\epsilon_0}{1+\chi_e} = \frac{\rho}{\epsilon_0(1+\chi_e)} = \boxed{\frac{\rho}{\epsilon} = \nabla\cdot\vec{E}}$$

valid if ϵ is indep. of position

So all the problems solved before are the same simply replacing ϵ_0 by ϵ !
(in the case in which the medium fills all space)
($\equiv \epsilon$ indep. of x)

If the medium does not fill all space, we must consider boundary conditions at the surface of contact:

$$\boxed{\begin{aligned}(\vec{D}_2 - \vec{D}_1)\cdot\vec{n}_{21} &= \sigma \\ (\vec{E}_2 - \vec{E}_1)\times\vec{n}_{21} &= 0\end{aligned}}$$



σ = surface charge density (not including polarization charge)

These boundary conditions are derived in page 16 of the book.