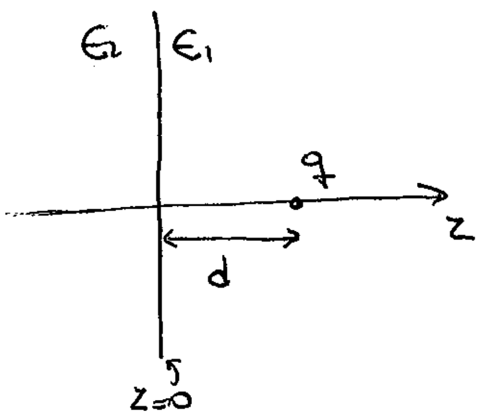
4.4 Boundary Problems

Method of images



$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_1} \quad z>0$$

point charge

$$\nabla \cdot \vec{E} = 0 \quad z<0$$

no charge

$$\nabla \times \vec{E} = 0$$

(4.27) This one is valid all the time

Boundary conditions at $z=0$:

$$(\vec{D}_2 - \vec{D}_1) \cdot \underbrace{\vec{n}_{21}}_{\vec{e}_z} = (\underbrace{D_{2z}}_{\epsilon_2 \underbrace{E_z^2}_{\lim\limits_{z\to 0^-} E_z}} - \underbrace{D_{1z}}_{\epsilon_1 \underbrace{E_z^1}_{\lim\limits_{z\to 0^+} E_z}}) = 0$$

From $\nabla \times \vec{E}$ and $(\vec{E}_2 - \vec{E}_1) \times \vec{n}_{21} = 0$ we get:

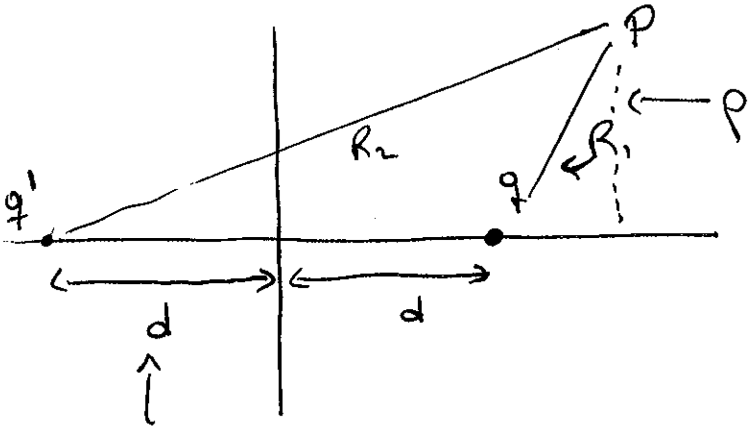
$$\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ (E_2-E_1)^x & (E_2-E_1)^y & (E_2-E_1)^z \\ 0 & 0 & 1 \end{vmatrix} = 0 \; ; \quad \begin{array}{l} (E_2-E_1)^y = 0 \\ (E_2-E_1)^x = 0 \end{array}$$

or

$$\lim_{z\to 0^-} E^y = \lim_{z\to 0^+} E^y$$

$$\lim_{z\to 0^-} E^x = \lim_{z\to 0^+} E^x$$

Let us use an image charge:



let us assume this distance is "d" as well and see if it works

$$\underset{\text{at point } P}{\Phi} = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) \qquad \underline{\underline{z > 0}}$$

This is the dielectric constant at point P, i.e. at the point we are investigating

$$R_1 = \sqrt{\rho^2 + (d-z)^2}$$

$$R_2 = \sqrt{\rho^2 + (d+z)^2}$$

Now, let us consider $\underline{\underline{z < 0}}$. On this side there are __no charges__, thus the only component to Φ will come from the "image charge" caused by $z > 0$.

However, at $z < 0$ the charge of the image is $\underline{\underline{\text{not}}}$ q but a new q" since we must deal with the influence of the boundary!

$$\underset{z<0}{\Phi} = \frac{1}{4\pi\epsilon_2}\frac{q''}{R_1}$$

Now let us use the boundary conditions:

$$\lim_{z\to 0^-} E^z = \lim_{z\to 0^+} E^z \quad , \text{etc}$$

$$\left.\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)\right|_{z=0} = \left(-\frac{1}{2}\right)\frac{1}{(\rho^2+(d-z)^2)^{3/2}}\,2(d-z)(-1)\Bigg|_{z=0} = \frac{d}{(\rho^2+d^2)^{3/2}}$$

$$\left.\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\right|_{z=0} = \left(-\frac{1}{2}\right)\frac{1}{(\rho^2+(d+z)^2)^{3/2}}\,2(d+z)\Bigg|_{z=0} = \frac{-d}{(\rho^2+d^2)^{3/2}}$$

$$\left.\frac{\partial}{\partial \rho}\left(\frac{1}{R_1}\right)\right|_{z=0} = \left(-\frac{1}{2}\right)\frac{1}{(\rho^2+(d-z)^2)^{3/2}}\,2\rho\Bigg|_{z=0} = \frac{-\rho}{(\rho^2+d^2)^{3/2}}$$

$$\left.\frac{\partial}{\partial \rho}\left(\frac{1}{R_2}\right)\right|_{z=0} = \left(-\frac{1}{2}\right)\frac{1}{(\rho^2+(d+z)^2)^{3/2}}\,2\rho\Bigg|_{z=0} = \frac{-\rho}{(\rho^2+d^2)^{3/2}}$$

From the z component: $\lim_{z\to 0^+} \epsilon_1 E_z^1 = \lim_{z\to 0^-} \epsilon_2 E_z^2$

$\vec{E} = -\nabla\phi$ comes sign

$$-\epsilon_1\frac{\partial}{\partial z}\left[\frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1}+\frac{q'}{R_2}\right)\right]\Bigg|_{z=0} = -\epsilon_2\frac{\partial}{\partial z}\left[\frac{1}{4\pi\epsilon_2}\left(\frac{q''}{R_1}\right)\right]\Bigg|_{z=0}$$

$$\epsilon_1\frac{q}{4\pi\epsilon_1}\left[-\frac{d}{(\rho^2+d^2)^{3/2}}\right] + \frac{q'}{4\pi\epsilon_1}\left[\frac{d}{(\rho^2+d^2)^{3/2}}\right] = \frac{-\epsilon_2 q''}{4\pi\epsilon_2}\cdot\frac{d}{(\rho^2+d^2)^{3/2}}$$

Then: $-q + q' = -q''$ or $\boxed{q - q' = q''}$

From the component perpendicular to the normal to the surface:

$$\lim_{z\to 0^-} E\rho = \lim_{z\to 0^+} E\rho$$

$$\frac{q}{4\pi\epsilon_1}\underbrace{\left[-\frac{\partial}{\partial\rho}\left(\frac{1}{R_1}\right)\right]}_{\frac{\rho}{(\rho^2+d^2)^{3/2}}} + \frac{q'}{4\pi\epsilon_1}\underbrace{\left[-\frac{\partial}{\partial\rho}\left(\frac{1}{R_2}\right)\right]}_{\frac{\rho}{(\rho^2+d^2)^{3/2}}} = \frac{q''}{4\pi\epsilon_2}\underbrace{\left[-\frac{\partial}{\partial\rho}\left(\frac{1}{R_1}\right)\right]}_{\frac{\rho}{(\rho^2+d^2)^{3/2}}}$$

Then

$$\boxed{\frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2}}$$

These two equations can be solved:

$$q'' = \underbrace{\frac{\epsilon_2}{\epsilon_1}(q+q') = q - q'}$$

$$\epsilon_2 q + \epsilon_2 q' = \epsilon_1 q - \epsilon_1 q'$$

$$q' = \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right) q = \boxed{-\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\right) q = q'}$$

$$q'' = q - q' = q - (-)\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\right) q = \left(\frac{\epsilon_1+\epsilon_2+\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\right) q =$$

$$= \frac{2\epsilon_2}{\epsilon_1+\epsilon_2} q$$

Let us calculate the polarization "jump":

$$\Delta P = P_z^2 - P_z^1 \Big|_{z=0} = \epsilon_0 \chi_e^2 E_z^2 - \epsilon_0 \chi_e^1 E_z^1 \Big|_{z=0} =$$

$$\underset{(4.38)}{=} \epsilon_0 \underbrace{\left(\frac{\epsilon_2}{\epsilon_0} - 1\right)}_{\chi_e^2} E_z^2 - \epsilon_0 \underbrace{\left(\frac{\epsilon_1}{\epsilon_0} - 1\right)}_{\chi_e^1} E_z^1 \Big|_{z=0}$$

From boundary condition used before,

$$\epsilon_1 E_z^1 = \epsilon_2 E_z^2 \quad \text{at } z=0$$

$$\text{or} \quad E_z^2 = \frac{\epsilon_1}{\epsilon_2} E_z^1$$

then,

$$\Delta P = \Bigg[\underbrace{\underbrace{\epsilon_0 \left(\frac{\epsilon_2}{\epsilon_0} - 1\right) \frac{\epsilon_1}{\epsilon_2}}_{(\epsilon_2 - \epsilon_0)\frac{\epsilon_1}{\epsilon_2}} - \underbrace{\epsilon_0 \left(\frac{\epsilon_1}{\epsilon_0} - 1\right)}_{\epsilon_1 - \epsilon_0}}_{\frac{\cancel{\epsilon_2\epsilon_1} - \epsilon_0\epsilon_1 - \cancel{\epsilon_1\epsilon_2} + \epsilon_0\epsilon_2}{\epsilon_2} = (\epsilon_2 - \epsilon_1)\frac{\epsilon_0}{\epsilon_2}} \Bigg] \underbrace{E_z^1 \Big|_{z=0}}_{\frac{(q + q') d}{4\pi\epsilon_1 (\rho^2 + d^2)^{3/2}}}$$

From previous pages:

$$\frac{(q + q')\, d}{4\pi\epsilon_1 (\rho^2 + d^2)^{3/2}}$$

and $q' - q = -q'' =$

$$= \frac{-2\epsilon_2}{\epsilon_1 + \epsilon_2} q$$ also from previous pages

then:

$$\Delta P = (\epsilon_2 - \epsilon_1)\frac{\epsilon_0}{\epsilon_2}\left(\frac{-2\epsilon_2}{\epsilon_1 + \epsilon_2}\right) q = -\epsilon_0 2 \left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\right) q \frac{1}{4\pi\epsilon_1} \frac{d}{(\rho^2 + d^2)^{3/2}}$$

$$\frac{1}{4\pi\epsilon_1} \cdot \frac{d}{(\rho^2 + d^2)^{3/2}} \Bigg| = \boxed{-\frac{q}{2\pi} \cdot \frac{\epsilon_0 (\epsilon_2 - \epsilon_1)}{\epsilon_1 (\epsilon_1 + \epsilon_2)} \cdot \frac{d}{(\rho^2 + d^2)^{3/2}}}$$

Note that from (4.33)

$$\nabla \cdot E = \frac{1}{\epsilon_0}\left(\rho - \nabla \cdot \vec{P}\right)$$

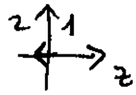
it is clear that $-\nabla \cdot \vec{P}$ is a "polarization charge density".

$-\nabla \cdot \vec{P}$ at the surface is proportional to $\vec{P}_2 - \vec{P}_1$ i.e. it is the "jump". Then:

$$\boxed{\sigma_{pol}} = -\left(\vec{P}_2 - \vec{P}_1\right) \cdot \underbrace{\vec{n}_{21}}_{} = \Delta P = P_2 - P_1 \Big|_{z=0} =$$

From 1 to 2

i.e. $-\vec{e}_z$



$$\boxed{= -\frac{q}{2\pi} \cdot \frac{\epsilon_0}{\epsilon_1} \frac{(\epsilon_2 - \epsilon_1)}{\epsilon_1 + \epsilon_2} \cdot \frac{d}{(\rho^2 + d^2)^{3/2}}}$$

<u>Limit</u>: Suppose $\epsilon_2 \to \infty$. Then, $\vec{E} = \frac{\vec{D}}{\epsilon_2} \to 0$

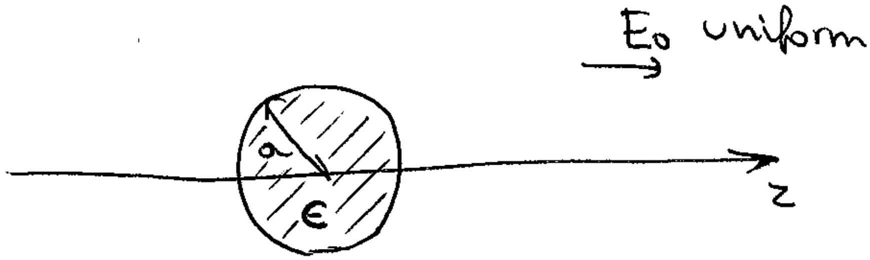
which is what happens in a conductor.

If $\epsilon_1 \to \epsilon_0$ then we have a problem as studied before i.e.

. Do the formulas agree?

$$\sigma_{pol} \xrightarrow[\epsilon_1 \to \epsilon_0]{\epsilon_2 \to \infty} \boxed{-\frac{q}{2\pi} \cdot \frac{d}{(\rho^2 + d^2)^{3/2}}}$$

[See Eq. (3.9) page 123 of Griffiths]



$\underline{\underline{\text{No}}}$ free charges inside or outside.

Thus, we have to solve the Laplace equation with the proper boundary condition. In other words:

$$\cancel{-\nabla^2\Phi = \nabla\cdot\vec{E} = \frac{1}{\epsilon_0}[\rho - \nabla\cdot\vec{P}] = \frac{1}{\epsilon_0}[\rho - (\epsilon-\epsilon_0)\nabla\cdot\vec{E}]}$$

$$\nabla\cdot\vec{D} = \underset{\uparrow\ \text{net charge}}{\rho} = 0 \quad \text{outside the surface}$$

But $\vec{D} = \epsilon\vec{E}$. Then, $\nabla\cdot\vec{E} = 0$ or $-\nabla^2\Phi = 0$.

Since the problem has uniaxial symmetry we use Eq. (3.33):

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

<u>Inside</u> sphere we have to discard the diverging $\underset{r\to 0}{r^{-(l+1)}}$

$$\Phi_{inside}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

<u>Outside</u>, we drop the $r\to\infty$ diverging r^l, <u>with the exception of $l=1$</u> since we want $\vec{E} \xrightarrow[|r|\to\infty]{} \vec{E^0}$

$$\Phi_{outside}(r,\theta) = \sum_{l=0}^{\infty}\left(B_l r^l + A_l r^{-(l+1)}\right) P_l(\cos\theta)$$

We take:

$$B_\ell = -E_0\,\delta_{\ell,1}$$

$$\underset{\text{outside}}{\Phi}(r,\theta) = \underbrace{-E_0 r P_1(\cos\theta)}_{-E_0 r\cos\theta = -E_0 z} + \sum_{\ell=0}^{\infty} C_\ell\, r^{-(\ell+1)} P_\ell(\cos\theta)$$

page 103 or 97

$P_1(\cos\theta) = \cos\theta$

$$\underset{\substack{\text{caused}\\ \text{by this}\\ \text{term}}}{\vec{E}} = -\nabla\phi = -\frac{\partial}{\partial z}(-E_0 z) = +E_0 \checkmark$$

Let us now use the boundary conditions:

We want the tangencial component of the electric field to be continuous based on (4.40). From the back of the book, pick the "e_2" component in $\nabla\psi$ "~~cylindrical~~ spherical" i.e. "$\frac{1}{r}\frac{\partial\psi}{\partial\theta}$", with $r=a$:

$$-\frac{1}{a}\left.\frac{\partial\Phi_{\text{inside}}}{\partial\theta}\right|_{r=a} = -\frac{1}{a}\left.\frac{\partial\Phi_{\text{outside}}}{\partial\theta}\right|_{r=a}$$

With regards to the normal component, it is the vector $\vec{D}$ that is continuous:

$$-\epsilon \frac{\partial \Phi_{inside}}{\partial r}\Big|_{r=a} \underset{\uparrow}{=} -\epsilon_0 \frac{\partial \Phi_{outside}}{\partial r}\Big|_{r=a}$$

$$\vec{D} = \epsilon \vec{E}$$

The first eq. gives:

$$-\frac{1}{a} \sum_{l=0}^{\infty} A_l a^l \frac{\partial}{\partial \theta} P_l(\cos\theta) = -\frac{1}{a}\Big[-E_0 a \frac{\partial}{\partial \theta} \underbrace{\cos\theta}_{P_1} + \sum_{l=0}^{\infty} C_l a^{-(l+1)} \frac{\partial}{\partial \theta} P_l(\cos\theta)\Big]$$

The coefficients must be equal term by term, thus:

$$-A_l a^{l-1} = E_0 \delta_{l,1} - C_l a^{-(l+2)}$$

or for $l=1$: $-A_1 = E_0 - C_1 a^{-3}$

$$\boxed{A_1 = -E_0 + \frac{C_1}{a^3}}$$

for $l \neq 1$: $-A_l a^{l-1} = -C_l a^{-(l+2)}$

$$\boxed{A_l = \frac{C_l}{a^{2l+1}}}$$

(4.51)

The second equation gives:

$$(-\epsilon) \cdot \sum_{l=0}^{\infty} A_l l \underbrace{a^{l-1}}_{P_l(\cos\theta)} = (-\epsilon_0)\Big[-E_0 \cos\theta + \sum_{l=0}^{\infty} C_l [-(l+1)] a^{-l-2} P_l(\cos\theta)\Big]$$

Since coeff. must be equal term by term:

$$-\epsilon A_\ell \ell a^{\ell-1} = (-\epsilon_0)\left[-E_0\delta_{\ell 1} + C_\ell[-(\ell+1)]a^{-\ell-2}\right]$$

For $\ell = 1$:

$$-\epsilon A_1 = (-\epsilon_0)\left[-E_0 - 2C_1\frac{1}{a^3}\right]$$

$$\boxed{\frac{\epsilon}{\epsilon_0}A_1 = -E_0 - 2\frac{C_1}{a^3}}$$

For $\ell \neq 1$:

$$-\epsilon A_\ell \ell a^{\ell-1} = (-\epsilon_0)\, C_\ell\, (-\ell-1)\, a^{-\ell-2} \qquad (4.52)$$

$$\boxed{\frac{\epsilon}{\epsilon_0}\ell A_\ell = -\frac{C_\ell}{a^{2\ell+1}}(\ell+1)}$$

The other boundary condition gave:

$$A_\ell = \frac{C_\ell}{a^{2\ell+1}}, \quad \ell \neq 1.$$

Putting all together we have

$A_\ell = \frac{C_\ell}{a^{2\ell+1}}$ on one hand and $-\frac{\epsilon}{\epsilon_0}\frac{\ell A_\ell}{(\ell+1)} = \frac{C_\ell}{a^{2\ell+1}}$ on the other.

They can be simultaneously right if $A_\ell = C_\ell = 0$ $\ell \neq 1$ or if $-\frac{\epsilon}{\epsilon_0}\frac{\ell}{(\ell+1)} = 1$ which is impossible.

Then, $\boxed{A_\ell = C_\ell = 0, \ \ell \neq 1}$

For $l=1$, we have:

$$A_1 = -E_0 + \frac{C_1}{a^3} \quad \text{and} \quad \frac{\epsilon}{\epsilon_0} A_1 = -E_0 - 2\frac{C_1}{a^3}$$

$$\frac{\epsilon}{\epsilon_0} A_1 = -E_0 - 2(A_1 + E_0) = -2A_1 - 3E_0$$

$$A_1 = \frac{-3E_0}{+2+\frac{\epsilon}{\epsilon_0}} = \boxed{-\left(\frac{3}{2+\frac{\epsilon}{\epsilon_0}}\right)E_0 = A_1}$$

Then:

$$\frac{C_1}{a^3} = A_1 + E_0 = -\frac{3}{2+\frac{\epsilon}{\epsilon_0}}E_0 + E_0$$

$$= \left(\frac{-3+2+\frac{\epsilon}{\epsilon_0}}{2+\frac{\epsilon}{\epsilon_0}}\right)E_0 = \left(\frac{\frac{\epsilon}{\epsilon_0}-1}{\frac{\epsilon}{\epsilon_0}+2}\right)E_0$$

$$\boxed{C_1 = \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right)a^3 E_0}$$

(4.53)

The potential is:

$$\Phi_{inside} = A_1 r^1 \underbrace{P_1(\cos\theta)}_{\cos\theta} = \boxed{-\left(\frac{3}{2+\frac{\epsilon}{\epsilon_0}}\right)E_0 r\cos\theta.}$$

(4.54)

$$\Phi_{outside} = (\underbrace{B_1}_{-E_0} r^1 + C_1 r^{-2})\underbrace{P_1(\cos\theta)}_{\cos\theta} = \boxed{-E_0 r\cos\theta + \left(\frac{\frac{\epsilon}{\epsilon_0}-1}{\frac{\epsilon}{\epsilon_0}+2}\right)\frac{E_0 a^3}{r^2}\cos\theta}$$

It is interesting that the potential inside the sphere describes a constant electric field parallel to the applied field of magnitude

$$E_{inside} = -\frac{d}{dz}\Phi_{inside} = \boxed{\frac{3}{\frac{\epsilon}{\epsilon_0}+2} E_0} \quad (4.55)$$

which $< E_0$ if $\epsilon > \epsilon_0$

Outside the sphere the potential has two terms. One is just the external uniform E_0 field. The second one has a $1/r^2$ dependence so it must be a dipole. From (4.10) we know a dipole is of the form $\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$ which is ~~over here~~ $\vec{p}$ points along z

would be $\frac{1}{4\pi\epsilon_0}\frac{p_z r\cos\theta}{r^3} = \frac{p_z}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$.

Comparing with $\Phi_{outside}$, second term, then

$$\boxed{p_z = 4\pi\epsilon_0\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) a^3 E_0} \quad (4.56)$$

This "$\vec{p}$" is located formally at the origin like all dipoles that represent a distribution of charge. The actual polarization $\vec{P}$ is

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \underset{\substack{\uparrow \\ 4.38}}{=} \epsilon_0\left(\frac{\epsilon}{\epsilon_0} - 1\right)\vec{E} = (\epsilon - \epsilon_0)\vec{E} =$$

inside i.e. (4.55)

$$= (\epsilon - \epsilon_0)\frac{3}{\left(\frac{\epsilon}{\epsilon_0}+2\right)}\vec{E}_0$$

$$\boxed{\vec{P} = 3\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right) \vec{E}_0} \qquad (4.57)$$

Note that if we integrate this result over the sphere we get

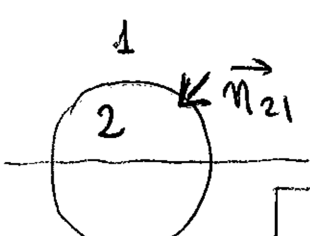
$$\vec{P} \times \mathrm{Vol}_{\text{sphere}} = \frac{4\pi}{3} a^3 \, 3\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right) \vec{E}_0$$

$$= \boxed{4\pi\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right) a^3 \vec{E}_0}$$

which is (4.56).

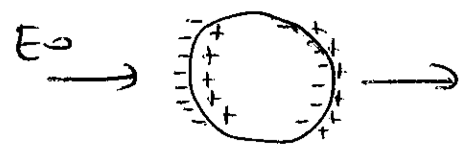
To get the "polarization-surface-charge" density we use $\sigma_{pol} = -(\vec{P}_2 - \vec{P}_1) \cdot \vec{n}_{21}$. From formulas such as $\vec{P} = \epsilon_0 \chi_e \vec{E} = \epsilon_0 \left(\frac{\epsilon}{\epsilon_0} - 1\right) \vec{E}$ we know that for $\epsilon = \epsilon_0$ i.e. outside then $\vec{P} = 0$.

Thus: $$\sigma_{pol} = -(\underset{\substack{\uparrow \\ (4.57)}}{\vec{P}} - \vec{0}) \cdot \underset{\text{inwards}}{\left(\frac{-\vec{r}}{r} \right)} = \frac{\vec{P} \cdot \vec{r}}{r}$$

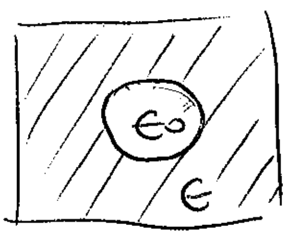


$$\boxed{\sigma_{pol} = 3\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right) \underbrace{\frac{\vec{E}_0 \cdot \vec{r}}{r}}_{\frac{E_0 \cancel{r} \cos\theta}{\cancel{r}}}} \qquad (4.58)$$

Effectively the problem inside the sphere is a superposition of the external field $\vec{E}_0$ and an internal field produced by the polarization with equal direction, different sign, such that the sum gives (4.55)



If we want to get the "inverse" problem of a spherical "cavity" inside a medium



it is enough to replace

$$\epsilon \to \epsilon_0$$
$$\epsilon_0 \to \epsilon$$

$$E_{inside} = \frac{3}{\frac{\epsilon_0}{\epsilon}+2} E_0 = \frac{3\epsilon}{\epsilon_0+2\epsilon} E_0$$

(4.57)

which is $> E_0$ if $\epsilon > \epsilon_0$. The field outside is again the sum of the constant $\vec{E}_0$ plus a field produced by a dipole. Note that p_z in (4.56) is $\propto \frac{\epsilon}{\epsilon_0} - 1$ so after replacing $\epsilon \to \epsilon_0$, $\epsilon_0 \to \epsilon$ we get $\frac{\epsilon_0}{\epsilon} - 1$ which is < 0, i.e. it points the other way.

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4.7 Electrostatic Energy in Dielectric Media

In free space, $\underset{\text{energy}}{W} = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$.

But this cannot be applied to a dielectric medium.
Let us derive better formulas. Consider a small change $\delta\rho(\vec{x})$ in the macroscopic charge density. The work needed will be

$$\delta W = \int \delta\rho(\vec{x})\,\Phi(\vec{x})\,d^3x.$$

Since $\nabla\cdot\vec{D} = \rho$, then $\delta\rho = \nabla\cdot(\delta\vec{D})$

$$\delta W = \underbrace{\int \nabla\cdot(\delta\vec{D})\,\Phi\,d^3x}_{\nabla\cdot(f\vec{v}) = \nabla f\cdot\vec{v} + f(\nabla\cdot\vec{v})} = -\int \underbrace{\nabla\Phi}_{-\vec{E}}\cdot\delta\vec{D}\,d^3x =$$

usual assumption of $\int \nabla\cdot(f\vec{v}) = 0$ valid if $\rho(\vec{x})$ is <u>localized</u>.

$$= \int \vec{E}\cdot\delta\vec{D}\,d^3x$$

$$\underset{\text{total energy}}{W} = \int d^3x \int_0^D \vec{E}\cdot\delta\vec{D}$$

For a <u>linear</u> medium, then $\vec{D} = \epsilon\vec{E}$

$$\boxed{\vec{E}\cdot\delta\vec{D} = \epsilon\,\vec{E}\cdot\delta\vec{E} = \frac{\epsilon}{2}(\vec{E}\cdot\delta\vec{E} + \delta\vec{E}\cdot\vec{E}) = \frac{\epsilon}{2}\,\delta(\vec{E}\cdot\vec{E}) = \frac{\delta(\vec{E}\cdot\vec{D})}{2}} \qquad (4.88)$$

$$\boxed{W = \frac{1}{2}\int d^3x \int_0^D \delta(\vec{E}\cdot\vec{D})} = \frac{1}{2}\int d^3x(\vec{E}\cdot\vec{D}) = \qquad (4.89)$$

$$= \frac{1}{2}\int d^3x\left[(-\nabla\phi)\cdot\vec{D}\right] = -\frac{1}{2}\int d^3x\left[\underbrace{\nabla\cdot(\phi\vec{D})}_{\to 0 \text{ if } \rho(\vec{x}) \text{ localized}} - \phi\underbrace{\nabla\cdot\vec{D}}_{\rho}\right] =$$

$$= \boxed{\frac{1}{2}\int d^3x\, \rho(\vec{x})\,\Phi(\vec{x})}$$

Which is the <u>same</u> formula for free space, but this was obtained assuming a <u>linear</u> medium with $\vec{D} = \epsilon\vec{E}$. If the medium is <u>not</u> linear then the relation is <u>not</u> valid.