Chapter 5

Section 5.1: Introduction

We will exploit the relation between currents and magnetic fields (since there are no "free magnetic charges"), instead of charges and electric fields.

Also in general the continuity equation says

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

which is just charge conservation. In this chapter we will consider a steady state where $\frac{\partial \rho}{\partial t} = 0$ and, thus,

$$\boxed{\nabla \cdot \vec{J} = 0}$$

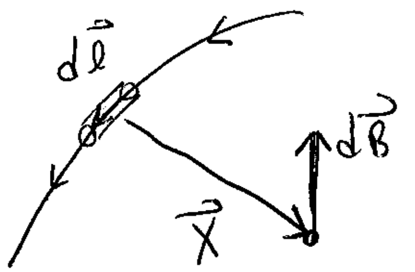
in magnetostatics.

[We will get back to a $\frac{\partial \rho}{\partial t} \neq 0$ in chapter 6 when we consider the Maxwell equations.]

5.2 Biot and Savart law

Basic relations: if $d\vec{l}$ is an element of length, pointing in the direction of current flow, of a wire that carries a current I, the $d\vec{B}$ is given by

$$d\vec{B} = k\, I\, \frac{(d\vec{l}\times\vec{x})}{|\vec{x}|^3} \qquad (5.4)$$

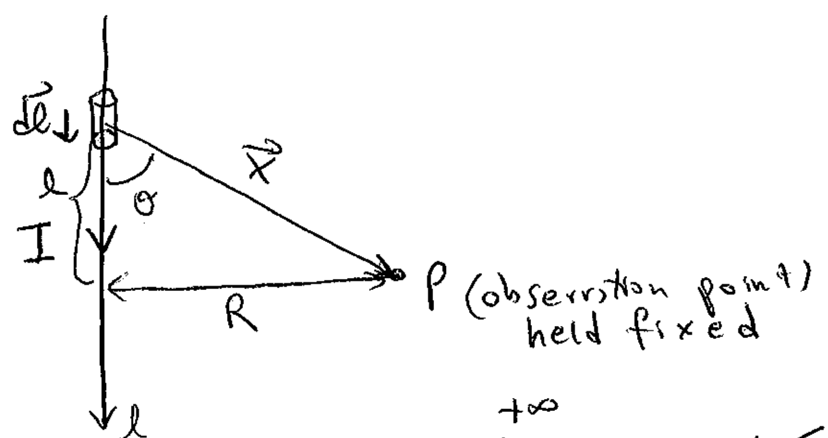


inverse square law as in Coulomb's force, but vector character is different.

$k = \frac{\mu_0}{4\pi}$ $\left(\frac{\text{Henry}}{\text{meter}}\right)$

$\boxed{\mu_0 \text{ is called "permeability of free space"} \quad \epsilon_0 \text{ was called "permittivity of free space"}}$

For a long wire (straight) we can integrate:

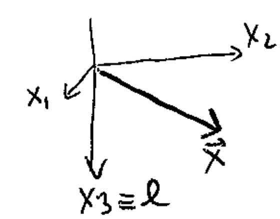


$$|\vec{B}| = \frac{\mu_0}{4\pi} I \int_{-\infty}^{+\infty} \frac{dl\, R}{(R^2+l^2)^{3/2}}$$

$|\vec{x}| = \sqrt{R^2+l^2}$

integrated over the wire

For $d\vec{l}\times\vec{x}$ consider coordinates as



and we can take $\vec{x} = (0, x_2, x_3)$ as special case without problems.

$d\vec{l} = (0, 0, dl)$

$$d\vec{l}\times\vec{x} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 0 & dl \\ 0 & x_2 & x_3 \end{vmatrix} = -dl\, x_2\, \vec{e}_1$$

$|d\vec{l}\times\vec{x}| = dl\, x_2 = dl\, R$

$$\int_{-\infty}^{+\infty} \frac{dl}{(R^2+l^2)^{3/2}} = \frac{2}{R^2} \quad \text{(left as exercise)}$$

Then: $$\boxed{|\vec{B}| = \frac{\mu_0}{2\pi}\frac{I}{R}}. \quad (5.6)$$

From (I.3), $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Lorentz force equation. It is the force acting on a point charge q due to fields $\vec{E}$ and $\vec{B}$.

Consider only a $\vec{B}$ field: $$d\vec{F} = \delta q(\vec{v} \times \vec{B}) = I(d\vec{l} \times \vec{B})$$

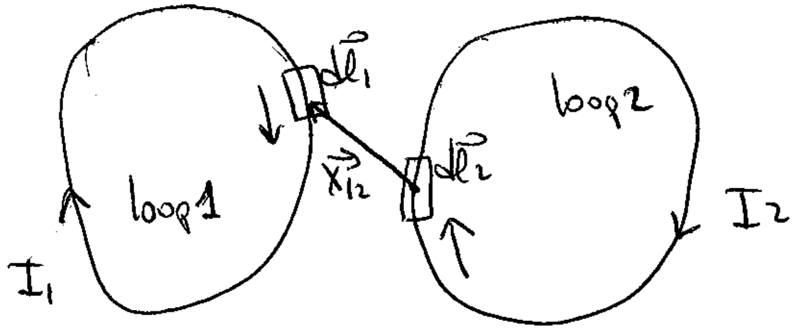
in our case current is charge in motion so $\delta q\, \vec{v} = \delta q \frac{d\vec{x}}{dt} = \underbrace{\frac{\delta q}{dt}}_{I} d\vec{x}$ in a steady state

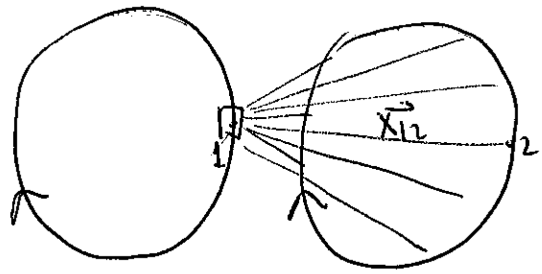
Suppose that $\vec{B}$ is due to a closed current loop "2". Then the force on current loop "1" will be:

$$\underset{\substack{\text{on } 1 \\ \text{caused} \\ \text{by } 2}}{\vec{F}_{12}} = \underset{\substack{\text{integral} \\ \text{over } 1}}{\oint} I_1 (d\vec{l}_1 \times \underbrace{\vec{B}_{\substack{\text{caused} \\ \text{by } 2}}}_{\frac{\mu_0}{4\pi} I_2 \oint \frac{(d\vec{l}_2 \times \vec{X}_{12})}{|\vec{X}_{12}|^3}})$$

$$= \boxed{\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{X}_{12})}{|\vec{X}_{12}|^3}} \quad (5.8)$$

More detail





Force at "dl_1" is proportional to total $\vec{B}$ created by loop 2

$$\frac{\mu_0}{4\pi} I_2 \oint_{\text{curve } 2} \frac{(d\vec{l}_2 \times \vec{X}_{12})}{|\vec{X}_{12}|^3}$$

and then we have to integrate this vector force over all loop 1

This force can be rewritten:

$$\underbrace{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{X}_{12})}_{\text{Identity: } \text{"}\vec{a}\times(\vec{b}\times\vec{c}) = -(\vec{a}\cdot\vec{b})\vec{c} + \vec{b}(\vec{a}\cdot\vec{c})\text{"}} = -(d\vec{l}_1 \cdot d\vec{l}_2)\vec{X}_{12} + d\vec{l}_2 (d\vec{l}_1 \cdot \vec{X}_{12})$$

The second term cancels, thus

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$$\frac{\vec{X}_{12}}{|\vec{X}_{12}|^3} = -\nabla\left(\frac{1}{|\vec{X}_{12}|}\right)$$

Thus, the second integral has $\oint_{\text{loop } 1} d\vec{l}_1 \cdot \nabla\left(\frac{1}{|\vec{X}_{12}|}\right)$ (It is the analog of a 1D integral of the form $\int dx \frac{dy}{dx} = \int dy$.)

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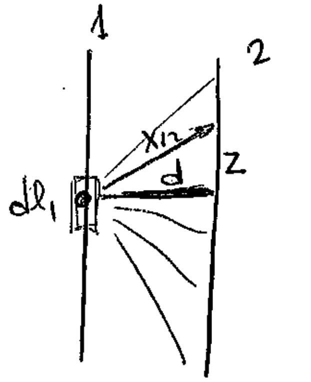
show that they cancel

$$\boxed{\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{(d\vec{l}_1 \cdot d\vec{l}_2)\vec{X}_{12}}{|\vec{X}_{12}|^3}} \quad (5.10)$$



Force at dl_1 caused by loop 2 is

$$|d\vec{F}| = \frac{\mu_0}{4\pi} I_1 I_2 \, dl_1 \underbrace{\int_{-\infty}^{+\infty} \frac{dZ\, d}{(Z^2+d^2)^{3/2}}}_{d \cdot \frac{2}{d^2}}$$

Integral used before

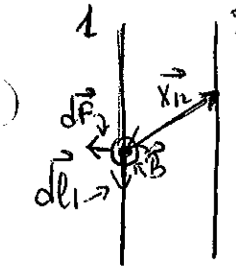
$X_{12} = \sqrt{d^2+Z^2}$

$|d\vec{l}_2 \times |\vec{X}_{12}|| = dz \cdot d$

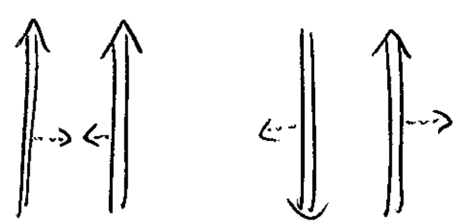
$$\boxed{\frac{|d\vec{F}|}{dl_1} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}} \qquad (5.11)$$

The force is attractive (repulsive) if the currents flow in the same (opposite) directions.

Left as exercise for students to see if they can do the vector algebra right.



$d\vec{F} \propto (d\vec{l}_1 \times \vec{B})$



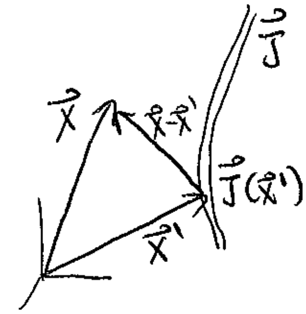
The formula $d\vec{F} = I_1 (d\vec{l}_1 \times \vec{B})$ can be written in more general terms as:

$$\boxed{\vec{F} = \int [\vec{J}(\vec{x}) \times \vec{B}(\vec{x})] \, d^3x} \qquad (5.12)$$

generates a $\vec{J}(\vec{x})$ not just in a thin wire.

by simply saying that $I_1 d\vec{l}_1$ could be considered a current density $\vec{J}(\vec{x})$ that has a $\vec{x}$ dependence instead of being a fixed current.

(5.4) $$d\vec{B} = \frac{\mu_0}{4\pi} I \left(\frac{d\vec{l} \times \vec{x}}{|\vec{x}|^3} \right)$$



Can be rewritten as $$d\vec{B} = \frac{\mu_0}{4\pi} \frac{J(\vec{x}) \times \vec{x}}{|\vec{x}|^3}$$

and

$$\boxed{\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int_{\text{all space}} J(\vec{x}') \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d^3x'}$$

This formula is very general but in practice it is not very useful.

Let us recall $$\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\nabla_x \left(\frac{1}{|\vec{x} - \vec{x}'|} \right)$$

(not x', but x)

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then:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \left[\nabla_x \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \right] \qquad (5.16)$$

$$\int \vec{J} \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = -\int \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \times \vec{J} = -\int \left(-\nabla \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \times \vec{J} \right)$$

$$= +\nabla \int \times \frac{\vec{J}}{|\vec{x} - \vec{x}'|}$$

$\vec{x}$-indep

$$\nabla \times (\varphi(x) \vec{A}) = \nabla \varphi(x) \times \vec{A}$$

It can be shown using cartesian coordinates and explicitly calculating both sides.

Then: $\nabla\cdot\vec{B} =$

$$= \frac{\mu_0}{4\pi}\,\underbrace{\nabla\cdot\left[\nabla\times\int\frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}\right]} = 0.$$

$$\nabla\cdot\underbrace{\nabla\times\vec{a}}$$

$$\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ \nabla_1 & \nabla_2 & \nabla_3\\ a_1 & a_2 & a_3\end{vmatrix} = \vec{e}_1(\nabla_2 a_3-\nabla_3 a_2) + \vec{e}_2(\nabla_3 a_1-\nabla_1 a_3) + \vec{e}_3(\nabla_1 a_2-\nabla_2 a_1)$$

$$\nabla\cdot(\;) = \nabla_1\cancel{\nabla_2 a_3}-\nabla_1\cancel{\nabla_3 a_2} + \nabla_2\cancel{\nabla_3 a_1}-\nabla_2\cancel{\nabla_1 a_3} + \nabla_3\cancel{\nabla_1 a_2}-\nabla_3\cancel{\nabla_2 a_1} = 0$$

Thus, the first equation of magnetostatics is

$$\boxed{\nabla\cdot\vec{B}=0}$$

Let us now calculate $\nabla\times\vec{B}$:

$$\nabla\times\vec{B} = \frac{\mu_0}{4\pi}\nabla\times\left(\nabla\times\int\frac{J(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x\right)$$

Use identity $\nabla\times(\nabla\times\vec{a}) = \nabla(\nabla\cdot\vec{a}) - \nabla^2\vec{a}$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \nabla \underbrace{\left(\nabla \cdot \int \frac{J(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \right)}$$

∇ acts on "x" only
thus this is

$$\int J(\vec{x}') \cdot \nabla \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) d^3x'$$

i.e.

$$\nabla \cdot \vec{a} f(x) =$$
$$= \nabla_1 a_1 f(x)$$
$$+ \nabla_2 a_2 f(x)$$
$$+ \nabla_3 a_3 f(x)$$
$$= a_1 \nabla_1 f(x) + \cdots$$
$$= \vec{a} \cdot \nabla f(x)$$

$$- \frac{\mu_0}{4\pi} \underbrace{\nabla^2 \int \frac{J(\vec{x}') d^3x'}{|\vec{x}-\vec{x}'|}}$$

∇ acts on "x" only
thus this is

$$\int \vec{J}(\vec{x}') \nabla^2 \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) d^3x'$$

$$= \frac{\mu_0}{4\pi} \nabla \int J(\vec{x}') \cdot \underbrace{\nabla \left(\frac{1}{|\vec{x}-\vec{x}'|} \right)} d^3x' \qquad \rightarrow -\nabla' \left(\frac{1}{|\vec{x}-\vec{x}'|} \right)$$

$$- \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \underbrace{\nabla^2 \left(\frac{1}{|\vec{x}-\vec{x}'|} \right)}_{-4\pi\delta(\vec{x}-\vec{x}')} d^3x'$$

$$= -\frac{\mu_0}{4\pi} \nabla \int \vec{J}(\vec{x}') \cdot \nabla' \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) d^3x' + \mu_0 \vec{J}(\vec{x})$$

We use now integration by parts: $\int d(uv) = \int u\,dv + \int v\,du$

$$\nabla' \cdot \left[\frac{J(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] = \overset{\text{only}}{\left[\nabla' \cdot J(\vec{x}')\right]} \frac{1}{|\vec{x}-\vec{x}'|} + J(\vec{x}') \cdot \nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

Then integrate and drop the ~~first~~ left hand side term as usual saying that it drops to zero at "∞". then

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \nabla \int \frac{\overset{\text{only}}{\left[\nabla' \cdot \vec{J}(\vec{x}')\right]}}{|\vec{x}-\vec{x}'|} d^3x' + \mu_0 \vec{J}$$

But in a steady state $\nabla \cdot \vec{J} = 0$ ("steady state" is the assumption of chapter 5)

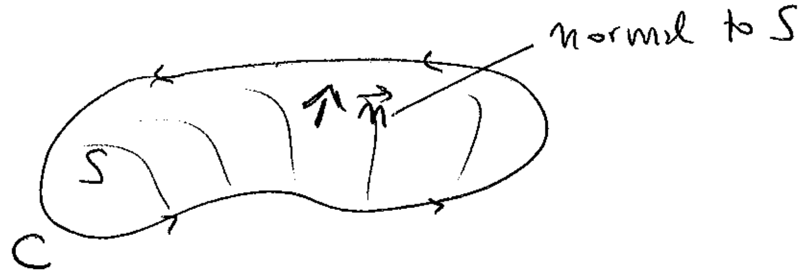
(while in general $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ (5.2))

then $\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$ (5.22)

So the two fundamental equations of magnetostatics are

$$\boxed{\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \mu_0 \vec{J} \end{aligned}}$$

We can construct an "integral" version of (5.22). Consider an open surface S bounded by a closed curve C.



$$\int_S (\nabla \times \vec{B}) \cdot \vec{n}\, da = \int_S \mu_0 \vec{J} \cdot \vec{n}\, dS = \oint_C \vec{B} \cdot d\vec{l} \qquad (5.23)$$

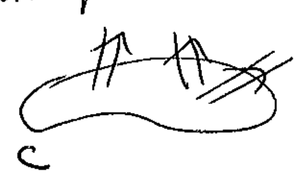
In general: $\int_S (\nabla \times \vec{F}) \cdot d\vec{a} = \int_C \vec{F} \cdot d\vec{l}$

i.e.

$$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \underbrace{\int_S \vec{J} \cdot \vec{n}\, da}_{\text{this is } \mu_0 I}} \qquad (5.24)$$

I: total current passing through the closed curve C.

$$(5.25) \qquad \boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I}$$



5.4 Vector Potential

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$
$$\nabla \cdot \vec{B} = 0$$

→ If $\nabla \cdot \vec{B} = 0$ <u>everywhere</u> (as it is) then $\vec{B}$ can be written as $\nabla \times \vec{A}$ ($\vec{A}$ = vector potential)

$$\vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}) \qquad (5.27)$$

In (5.16) we already found $\vec{B}$ as $\nabla \times$ of a vector. Thus:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' + \underbrace{\nabla \psi(\vec{x})} \qquad (5.28)$$

We can always add a gradient of an arbitrary scalar function ψ since $\underbrace{\nabla \times \nabla}\psi \equiv 0$

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 \end{vmatrix} = e_1 \underbrace{(\partial_2\partial_3 - \partial_3\partial_2)}_{=0} + \cdots$$

The extra term means that $\vec{A}$ is defined up to a $\nabla\psi(\vec{x})$, or that we can choose $\nabla\psi(\vec{x})$ as we like in order to simplify a problem.

Take the operation $\nabla \cdot$ on both sides:

$$\nabla \cdot \vec{A} = \frac{\mu_0}{4\pi} \underbrace{\int \vec{J}(\vec{x}') \cdot \nabla \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x'}_{\equiv -\int \vec{J}(\vec{x}') \cdot \nabla' \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x'} + \nabla^2 \psi(\vec{x})$$

Integrating by parts as we did a few lines before and we get: $\frac{\mu_0}{4\pi}\int \frac{(\nabla' \cdot \vec{J}(\vec{x}'))}{|\vec{x}-\vec{x}'|} d^3x' = 0$

in steady state $\nabla' \cdot \vec{J} = 0$

Then: $\nabla \cdot \vec{A} = \nabla^2 \psi(\vec{x})$ = only function of ψ, not $\vec{J}$.

If we choose $\nabla \cdot \vec{A} = 0$ then $\nabla^2 \psi(\vec{x}) = 0$ in all space. If there are no sources at infinity then $\psi(\vec{x})$ = constant and $\nabla\psi = 0$.

Then, with $\boxed{\nabla \cdot \vec{A} = 0}$ goes

called "Coulomb gauge" It will be discussed again in Chapter 6.

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'} \quad (5.32)$$

In general, the other fundamental equation is

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$$

$$\underbrace{\nabla(\nabla \cdot \vec{A})}_{=0 \text{ in Coulomb gauge}} - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

(5.31)

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$ if $\nabla \cdot \vec{A} = 0$ is chosen