6.4 Green Functions for the Wave Equation

Important for Chapter 9

Eq. for $\vec{A}$ in Coulomb gauge

Equations (6.15), (6.16), (6.30) are all of the form

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{x},t). \qquad (6.32)$$

can be $\rho(\vec{x},t)$ or $\vec{J}(\vec{x},t)$ and it is supposed to be given.

$$\boxed{\begin{aligned}\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -\rho/\epsilon_0 \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\vec{J}\end{aligned}}$$

To solve this type of equation it is useful to find a Green function, as we did in electrostatics. We will assume for simplicity that there are no boundary surfaces.

First, we define the Fourier transformed:

$$\psi(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

$$\frac{\partial^2\psi}{\partial t^2}(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\vec{x},\omega)\,(-\omega^2)\, e^{-i\omega t}\, d\omega \qquad (6.33)$$

We also define:

$$f(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

and the inverse relations shown in (6.34)

Then, we get:

$$\Big[\nabla^2 \underbrace{- \frac{1}{c^2}(-\omega^2)}_{+k^2}\Big]\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega)$$

$\left(k = \frac{\omega}{c}\right)$

Helmholtz equation:

$$\boxed{(\nabla^2 + k^2)\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega)} \qquad (6.35)$$

where ω is a parameter i.e. ∇^2 acts on $\vec{x}$ only and k^2 is a number, like wavector associated to ω

Similar to Poisson Eq. (equal if $k^2 = 0$)

The Green function $G(\vec{x},\vec{x}')$ for (6.35) is defined via:

$$(\nabla^2 + k^2)G_k(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}'). \qquad (6.36)$$

Assuming there are no surfaces, $G_k(\vec{x},\vec{x}')$ must depend on $\vec{x}-\vec{x}'$ only and actually by spherical symmetry it must depend only on $R = |\vec{R}| = |\vec{x}-\vec{x}'|$.

Using laplacian in spherical coordinates and using $G_k = G_k(R)$, (back cover)
we get:

$$\frac{1}{R}\frac{d^2}{dR^2}(R G_k) + k^2 G_k = -4\pi\,\delta(\vec{R}) \qquad (6.37)$$

If $\vec{R} \neq \vec{0}$, then:

$$\frac{1}{R}\frac{d^2}{dR^2}(R G_k) + k^2 G_k = 0$$

$$\frac{d^2}{dR^2}(RG_k) + k^2 (RG_k) = 0$$

Solution: $\boxed{RG_k = A e^{ikR} + B e^{-ikR}}$

$$\frac{d(RG_k)}{dR} = A\,ik\,e^{ikR} + B(-ik)e^{-ikR}$$

$$\frac{d^2}{dR^2}(RG_k) = A(ik)^2 e^{ikR} + B(-ik)^2 e^{-ikR}$$

$$= -k^2 (RG_k) \qquad \text{O.K.}$$

As $R \to 0$, the influence of the δ function grows as compared to the constant "k^2" term. Then from $(\nabla^2 + k^2)G = -4\pi\delta$

we get $\nabla^2 G = -4\pi\delta$ (1.39) and $G = \frac{1}{R} + F$. Thus at small R, $G \sim \frac{1}{R}$.

$$G_k = A \underbrace{\frac{e^{ikR}}{R}}_{\text{def. } G_k^{(+)}(R)} + B \underbrace{\frac{e^{-ikR}}{R}}_{\text{def. } G_k^{(-)}(R)} \qquad (6.39) \quad (6.41)$$

For $G_k \xrightarrow[R\to 0]{} \frac{1}{R}$ we must have $\boxed{A+B = 1.}$

Since $k = \frac{\omega}{c}$, then

e^{ikR} is a diverging spherical wave

e^{-ikR} is a converging spherical wave

This can only be understood if $e^{-i\omega t}$ is added.

A, B depend on boundary conditions in time

(remember we assume there are no real surfaces in the problem)

If a source is "switched on" at $t=0$, then e^{ikR} is the correct one, but a longer description is needed.

Let us construct the new G functions, "time dependent Green functions" that satisfy:

$$\boxed{\left(\nabla_x^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^{(\pm)}(\vec{x},t;\vec{x}',t') = -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')} \qquad (6.41)$$

From (6.34):

$$f(\vec{x},\omega) = \int_{-\infty}^{+\infty} \underbrace{f(\vec{x},t)}_{-4\pi\delta(\vec{x}-\vec{x}')\delta(t-t')} e^{i\omega t}\,dt = -4\pi\,\delta(\vec{x}-\vec{x}')\,e^{i\omega t'}$$

Then, the analog of (6.36) becomes:

$$\boxed{(\nabla^2 + k^2)\,G^{(\pm)}(R,\omega) = -4\pi\delta(\vec{x}-\vec{x}')\,e^{i\omega t'}}$$

~~The F. transf. of $G(\vec{x},t,\vec{x}',t')$ involve $e^{-i\omega(t-t')}$~~

Note that in time there is no translat. invariance like in $\vec{x}-\vec{x}'$, i.e. t and t' are very different: for instance the sources can be switched on at t=0.

which is the same eq. as found before in (6.36) if

$$G^{(\pm)}(R,\omega)\,e^{-i\omega t'} = G_k^{(\pm)}(R)$$

Then,

$$G_k^{(\pm)}(R,\omega) = \underbrace{G_k^{(\pm)}(R)}_{\frac{e^{\pm ikR}}{R}}\,e^{i\omega t'}$$

and doing the F.T. (6.33)

$$\boxed{G^{(\pm)}(R,t-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{\frac{e^{\pm ikR}}{R}\,e^{i\omega t'}}_{G_k^{(\pm)}(R,\omega)}\,e^{-i\omega t}\,d\omega} \qquad (6.42)$$

$e^{i\omega t'}e^{-i\omega t} = e^{-i\omega\tau}$, $\tau = t-t'$

Then, $G^{(\pm)}(R,\tau)$ is only a function of R and $\tau = t - t'$.

Note that the integral

see (2.46)

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pm ikR}\, e^{-i\omega(t-t')}\, d\omega = \delta\left(\pm\frac{R}{c} - (t-t')\right) =$$

$k = \frac{\omega}{c}$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(k-k')x}\, dx = \delta(k-k') \quad (2.46)$$

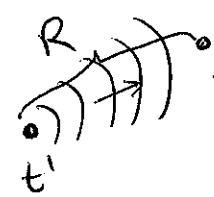
$$= \delta\left(t' - \left(t \mp \frac{\overbrace{|x-x'|}^{R}}{c}\right)\right)$$

$$\boxed{G^{(\pm)}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left[t' - \left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right]} \qquad (6.44)$$

"+" is called the "retarded Green function" because it shows the causal behavior associated with a wave disturbance i.e. $t' - t + \frac{R}{c} = 0$

$$\boxed{t' = t - R/c}$$

An effect observed at time "t" is caused by a source at R at an earlier time "t'" (or retarded time t') R/c is the time it takes the disturbance to travel from one point to another



$t = t' + R/c$

$G^{(-)}$ is the "advanced" Green function

In page 39, when G functions were discussed in the context of electrostatics, it was said in (1.42) that the solution of the Poisson Eq. $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$ was $\Phi = \frac{1}{4\pi\epsilon_0}\int \rho\, G\, d^3x$ (if surface terms are not present, as in our present study)

$$= 1 \cdot \int \left(\frac{\rho}{4\pi\epsilon_0}\right) G\, d^3x$$

Our eq. to solve is $(\nabla^2+k^2)\psi = -4\pi f$

Then, ρ is, formally, like "$4\pi f \epsilon_0$" and thus:

$\psi = \int G f$ or more explicitly

$$\boxed{\psi^{(\pm)}(\vec{x},t) = \iint d^3x'\,dt'\; G^{(\pm)}(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')}$$

Since $G^{(\pm)} = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left[t' - \left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right]$, we know $\psi^{(\pm)}$.

Which is more reasonable, "+" or "−"? Consider a source f localized in <u>both</u> space and time. If f is "on" say at a particular time $t' = t_0$, then the important times "t" must be <u>later</u> i.e. $t = t_0 + R/c$, they can't be before due to causality.

Then, the important one is:

$$\psi^{(+)}(\vec{x},t) = \iint d^3x'\,dt'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\delta\!\left(t'-\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)\right) f(\vec{x}',t')$$

and dropping "(+)":

(6.47)

$$\boxed{\psi(\vec{x},t) = \int d^3x'\,\frac{f(\vec{x}',t')_{ret}}{|\vec{x}-\vec{x}'|}}$$

where $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$

(this is the meaning of "ret")

In particular:

$$\boxed{\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\left[\vec{J}(\vec{x}',t')\right]_{ret}} \quad (6.48)$$

We will use this in Chapter 9 for "radiation"

Comparing (6.16) or (6.30) with (6.32), then

$4\pi f \equiv \mu_0 J$ so $f \equiv \frac{\mu_0}{4\pi} J$, thus the $\frac{\mu_0}{4\pi}$ in front on (6.48).