The electric field is given by

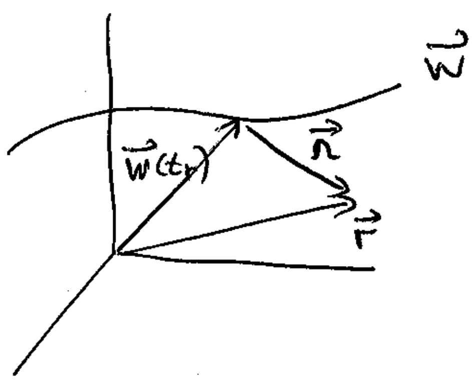
$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\cdot\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\cdot\left[\vec{u}(c^2-v^2)+\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]}$$

(9.107)
Griffiths.

where

$\vec{u} = c\hat{\mathscr{r}} - \vec{v}$ (9.106)

$\vec{v}, \vec{a}$ evaluated at retarded time



$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

$\vec{a}$ = acceleration at time retarded

<u>Example 5</u> of Griffiths (page 421, 2nd Ed.)

For the particular case of $\vec{a}=0$, then $\vec{w} = \vec{v}t$

$$\mathscr{r}\vec{u} = c\vec{\mathscr{r}} - \mathscr{r}\vec{v} = c(\vec{r}-\vec{v}t_r) - \mathscr{r}\vec{v} = c(\vec{r}-\vec{v}t)$$

$\mathscr{r} = c(t-t_r)$

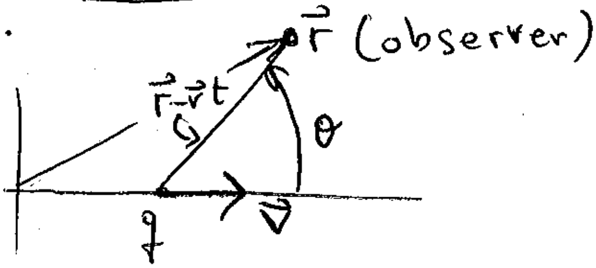
Note that "t_r" is gone and "t" replaces it.

etc. It can be shown that

$\vec{r}-\vec{v}t$ is a vector from the "observer" at $\vec{r}$ and the position of the charge at t, <u>not</u> at t_r, which is remarkable (quite special case).

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\,\frac{(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3}\,\frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}}$$

where



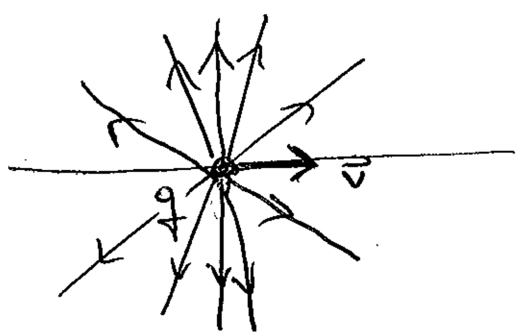
$\vec{v}$ = constant

The electric field has a θ dependence.

For $\theta = \pi/2$, the denominator is the smallest, thus the magnitude of $\vec{E}$ is <u>enhanced</u> by a $\frac{1}{\sqrt{1-v^2/c^2}}$ factor.

For $\theta = 0$, the $\vec{E}$ is the smallest, and its magnitude is suppressed by a $(1-v^2/c^2)$ factor

We represent this as more lines of $\vec{E}$ field in one direction than another:



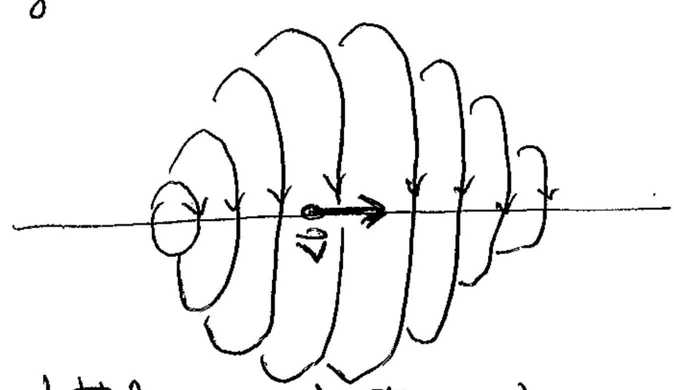
"Strength" goes like $\frac{1}{\text{distance}^2}$

So it is <u>not</u> radiating, since $\vec{a} = 0$.

Since $\vec{v}$ (vector) breaks rotational invariance, it is not too surprising that the magnitude of $\vec{E}$ depends on θ. Isotropic, like in the static limit, would be unnatural.

$\vec{B} = \frac{1}{c^2}(\vec{v} \times \vec{E})$ and this gives

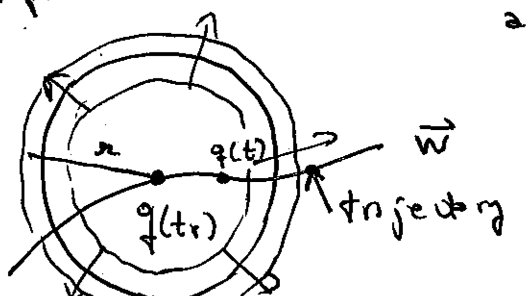
it can be shown



It is like a little current I and Ampere's law.

Power radiated by a point charge

We can calculate the Poynting vector $\vec{S}$ since we know $\vec{E}$ and $\vec{B}$ for a point particle that moves with velocity $\vec{v}$ and acceleration $\vec{a}$ (formulas 9.107 and 9.108 of Griffiths, 2nd edition). The energy flux that we can calculate via $\vec{S}$ contains just "field energy carried along by the particle as it moves" (like a polaron that distorts a lattice) and also "radiated energy" that detaches from the charge and propagates to infinity. Thus, to calculate power radiated we need to consider a huge sphere and wait a time so that the radiated energy reaches the sphere (so the particle may be at $\vec{w}(t)$, but what arrives to the surface of the huge sphere is what originated at $\vec{w}(t_r)$)



Only the $\vec{E}$ and $\vec{H}$ such that $\vec{S}$ goes like $1/\mathscr{r}^2$ needs to be considered, the rest do not contribute. Thus, only the acceleration field matters.

We already know that $\vec{S}_{rad} = \dfrac{1}{\mu_0 c} \vec{E}_{rad}^2 \hat{\mathscr{r}}$

(for instance from plane wave analysis)

and $$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left(\vec{\mathscr{r}} \times (\vec{u} \times \vec{a})\right)$$

$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

$\vec{a}$ = acceleration at t_r

$\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

Special case

If the charge is at rest $\vec{v}=0$ at a given retarded time but it is accelerating, then it can be shown that

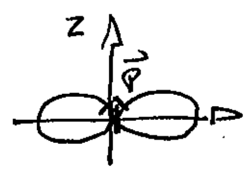
$$\vec{S} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{4\pi c^3} \cdot \frac{a^2 \sin\theta}{\mathscr{r}^2}\, \hat{\mathscr{r}} \qquad (9.115)$$

Poynting

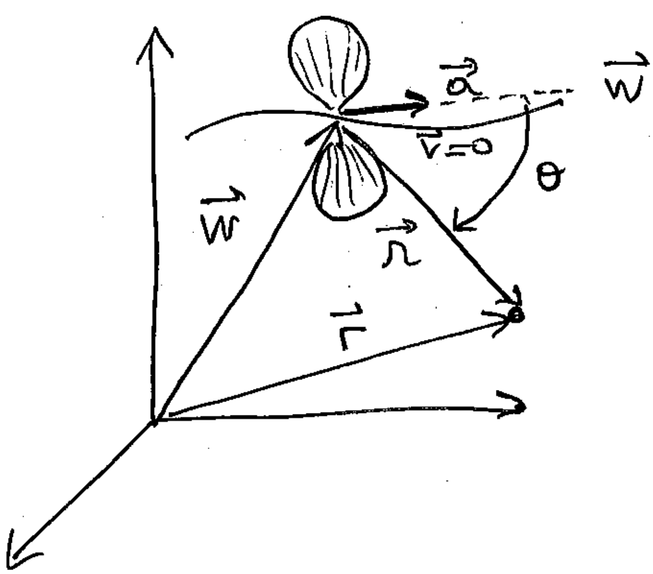
At $\theta=0$, no power is radiated: if we are along the line of $\vec{a}$, we do not get radiation.

$$\underset{\vec{v}=0}{\text{- - -}\bullet\!\!\longrightarrow} \vec{a} \text{ - - - - -} \times \text{- no radiation}$$

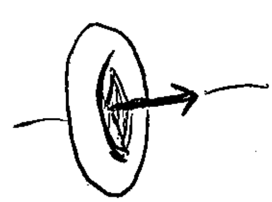
At $\theta=\pi/2$ it is maximized. This is similar to an electric dipole:



so a pair (+) (−) of oscillating charges mathematically behaves like an accelerating particle.



A better drawing is a donut:



for $\vec{v}=0$, $a \neq 0$

The total power radiated is given by

same as $\mathcal{P} = \int \frac{d\mathcal{P}}{d\Omega} d\Omega$ as brem some previous cases

$$\mathcal{P} = \int \vec{S}_{rad} \cdot d\vec{s} =$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{4\pi c^3} \iint \frac{a^2 \sin^2\theta}{\cancel{r^2}} \hat{r} \cdot \cancel{r^2} \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{4\pi c^3} a^2 \, 2\pi \underbrace{\int_0^\pi \underbrace{\sin^2\theta}_{1-\cos^2\theta} \sin\theta \, d\theta}$$

$$\underbrace{\int_0^\pi \sin\theta \, d\theta} - \int_0^\pi \sin\theta \cos^2\theta \, d\theta$$

$$-\cos\theta \Big|_0^\pi + \frac{\cos^3\theta}{3} \Big|_0^\pi$$

$$= 2 - \frac{2}{3} = \frac{4}{3}.$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \underbrace{\frac{4}{3} \frac{2\pi}{4\pi}}_{2/3} \frac{q^2 a^2}{c^3}$$

$$\boxed{\mathcal{P} = \frac{1}{4\pi\epsilon_0} \cdot \frac{2}{3} \cdot \frac{q^2 a^2}{c^3}}$$

(9.116) Griffiths 2nd edition

This is the Larmor formula

(See also (14.22) of Jackson)

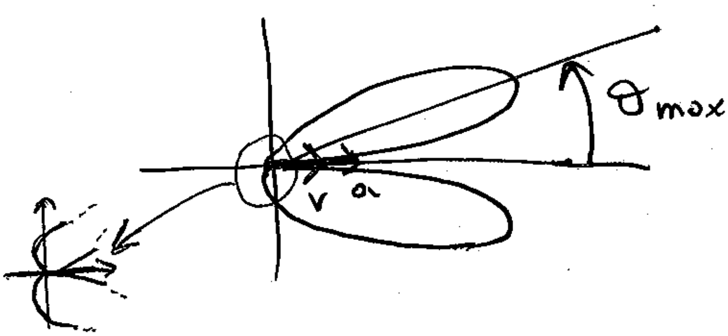
Another special case

If $\vec{v}$ and $\vec{a}$ are both nonzero at the retarded time and they are collinear, we can get a simple formula (9.120, Griffiths):

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{4\pi c^3} \cdot \frac{a^2 \sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^5}},$$

For the particular case $v = 0$, we recover the previous result i.e. the "donut"  $(\vec{v} = 0)$.

For $v \neq 0$, it is no longer a donut. Actually the factor $\frac{1}{\left(1-\frac{v}{c}\cos\theta\right)^5}$ influences the results very much particularly at large v: as soon as θ is nonzero, the denominator influences a lot and we get



The total power is

$$\boxed{P = \frac{1}{4\pi\epsilon_0} \cdot \frac{2}{3} \cdot \frac{q^2 a^2}{c^3} \gamma^6},$$

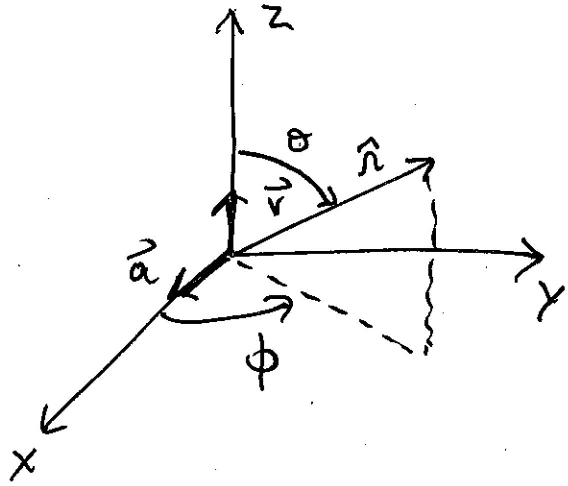
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

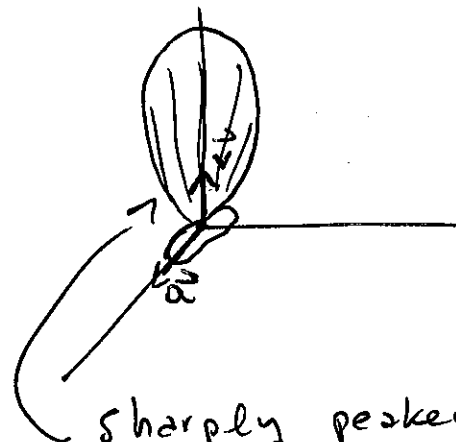
(see also (14.43) Jackson)

Note that the result of the "deformed donut" is the same for "a" possitive or negative one. It is the same whether the particle is <u>accelerating</u> or <u>decelerating</u>. When a fast electron hits a target and abruptly is stopped, it radiates with the deformed donut shape. This is called "braking radiation" or "bremsstrahlung".

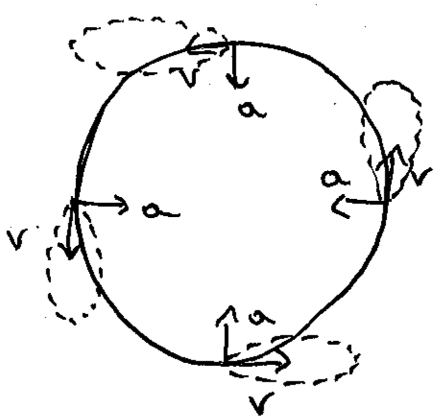
Other special cases

If $\vec{v}$ and $\vec{a}$ are perpendicular, the result is also interesting:





The most important application is for circular motion. In this case the radiation is called "synchrotron radiation". In this case the radiation sweeps like a locomotive's head light as the electron circles.



this is a big source of power dissipation and a problem in circular accelerator such as those at CERN.

(see Jackson, pages 667-668)

Since we have all the information at hand, we can also calculate P for the case in which the particle has a velocity $|\vec{v}| \neq 0$, and $\vec{v}, \vec{a}$ not collinear

The answer is:

$$\boxed{P = \frac{1}{4\pi\epsilon_0} \cdot \frac{2}{3} \cdot \frac{q^2}{c^3}\, \gamma^6 \left[a^2 - \left|\frac{\vec{v}}{c} \times \vec{a}\right|^2\right]} \quad *$$

where $\gamma^{-1} = \sqrt{1 - \frac{v^2}{c^2}}$

If $\vec{v} = 0$, $\gamma = 1$ and we recover the Larmor formula.

If $|\vec{v}|$ is very large, close to c, then γ^6 is very large: the radiated particle power by the particle increases enormously as $|\vec{v}|$ approaches c,

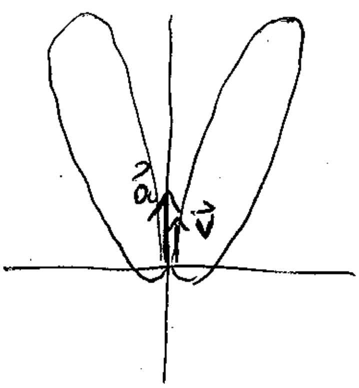
but still we need a nonzero acceleration.

If $\vec{v}, \vec{a}$ are collinear, then $\vec{v} \times \vec{a} = 0$ and we recover a previously discussed result.

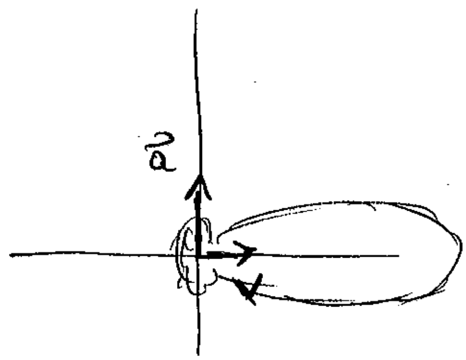
* see also (14.26) Jackson

In Griffiths (9.118) (2nd edition) we have the formula for $\frac{dP}{d\Omega}$ in general. But we can speculate on the radiation profile for an arbitrary angle between $\vec{a}$ and $\vec{v}$.

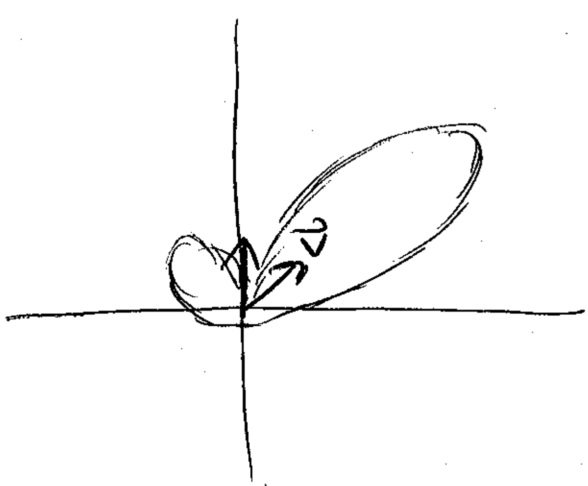
For $\vec{a} \parallel \vec{v}$:



For $\vec{a} \perp \vec{v}$:



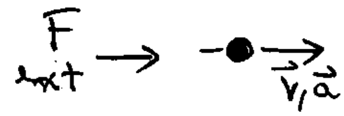
For an arbitrary angle my guess is:



Radiation Reaction

(See also Jackson 16.2)

A charge that is accelerating is known to radiate. Radiation carries off energy, which is given by the particle's kinetic energy.



but

Then, it is as if there is another force F_{rad}

$$F_{ext} + F_{rad}^{*} = m a .$$

Let us make a crude estimation of F_{rad}.

From previous calculations we know that a particle with acceleration "a" emits a total power (energy/time)

(9.122) Larmor formula →
$$\mathcal{P} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2 a^2}{c^3}$$
in the limit of small velocity.

$$\mathcal{P} = \frac{\Delta E}{\Delta t} \quad ; \quad \text{But } \Delta W_{work} = + \underbrace{\vec{F}_{rad} \cdot \vec{\Delta x}}_{\text{force . distance}} = - \Delta E$$

in general $\text{Work}_{i \to f} = -\Delta E = E_i - E_f$

$$\frac{\Delta E}{\Delta t} = - \vec{F}_{rad} \cdot \underbrace{\frac{\vec{\Delta x}}{\Delta t}}_{\vec{v}} = \mathcal{P}$$

$$\vec{F}_{rad} \cdot \vec{v} = - \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2 a^2}{c^3} \qquad (9.123)$$

* It is like a "recoil force": If you emit something with a finite momentum, its conservation says that you must loose some momentum.

Strictly speaking the energy lost as radiation does not happen instantaneously but first the particle gives ~~charge~~ energy to the immediate surrounding and then eventually that energy goes to infinity.

If the particle is considered in a time interval (t_1, t_2) where it returns to its initial state, then the energy lost is only radiation.

$$\int_{t_1}^{t_2} \vec{F}_{rad}\cdot\vec{v}\,dt = -\frac{1}{4\pi\epsilon_0}\cdot\frac{2}{3}\frac{q^2}{c^3}\int_{t_1}^{t_2} a^2\,dt$$

An example is a periodic motion.

Now: $$\int_{t_1}^{t_2} a^2\,dt = \int_{t_1}^{t_2} \underbrace{\frac{d\vec{v}}{dt}}_{"v"}\cdot\underbrace{\frac{d\vec{v}}{dt}}_{"du"}\,dt = -\int_{t_1}^{t_2}\underbrace{\vec{v}}_{"u"}\cdot\underbrace{\frac{d^2\vec{v}}{dt^2}\,dt}_{"dv"} + \underbrace{\vec{v}}_{"u"}\cdot\underbrace{\frac{d\vec{v}}{dt}}_{"v"}\Bigg|_{t_1}^{t_2}$$

$$\int v\,du = -\int u\,dv + uv$$

$$= -\int_{t_1}^{t_2}\underbrace{\dot{\vec{a}}}_{\frac{d^2\vec{v}}{dt^2}}\cdot\vec{v}\,dt + 0$$

Then:

$$\int_{t_1}^{t_2} \underbrace{\left[\vec{F}_{rad} - \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \dot{\vec{a}} \right]}_{\text{the Eq. is indeed equal to 0 if this is 0.}} \cdot \vec{v}\, dt = 0 \qquad (9.125)$$

$$\boxed{\vec{F}_{rad} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \dot{\vec{a}}} \qquad (9.126)$$

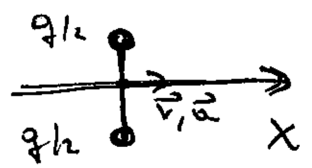
Abrahams-Lorentz formula for the radiation reaction force.

In 9.3.2 there is a complicated discussion based on giving a finite size to the particle.

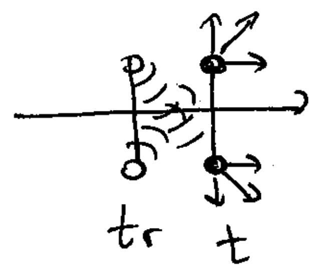
This is a force caused by the reaction of the charged particle to its own radiation field. It is like a "self-force". If the particle were to have a finite size, then it is understandable that, say, 1/2 of the particle sends radiation to the other half:

. The amazing thing is that this self-force survives even for a pointlike particle!

To explain this in a bit more detail
Consider a charge as two halves separated by distance d
moving in the x direction



At a later time, each half receives the effect of the other half at time tretarded.



By symmetry the $\perp$ forces will cancel, but along the x axis they add up! (see sketch)

Then, there is a force on itself.

The most remarkable thing is that this calculation survives the $d \to 0$ limit (as shown in Griffiths)

See also Chapter 16 of Jackson.

This topic is too complicated and advanced. This is a good moment to arrive to the end of the semester!

Note: Consider a charged particle attached to a spring, in the presence of a "driving force" $q E_0 \cos(\omega t)$:

$$m\ddot{x} = F_{spring} + F_{radiation} + F_{driving}$$

$$= -m\omega_0^2 x + m\tau\dddot{x} + q E_0 \cos(\omega t)$$

$$x = x_0 \cos(\omega t + \delta) \quad \leftarrow \text{in the steady state}$$

$$\dot{x} = x_0 \sin(\omega t + \delta)(-\omega)$$

$$\ddot{x} = x_0 \cos(\omega t + \delta)\,\omega^2$$

$$\dddot{x} = x_0 \sin(\omega t + \delta)(-\omega^3) = -\omega^2 \dot{x}$$

$$m\ddot{x} + m\omega_0^2 x + m\underbrace{\omega^2\tau}_{\gamma}\dot{x} = F_{driving}$$

plays the role of a damping factor caused by the radiation emitted by the accelerated particle

$m\tau\dddot{x}$ is $\frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c^3}\dot{a}$, then $\tau = \frac{1}{m}\cdot\frac{1}{4\pi\epsilon_0}\cdot\frac{2}{3}\cdot\frac{q^2}{c^3}$

Thus, when in past lectures we assumed a damping force proportional to the velocity, one of its origins is the fact that a charge radiates i.e. the coupling to the E&M field.