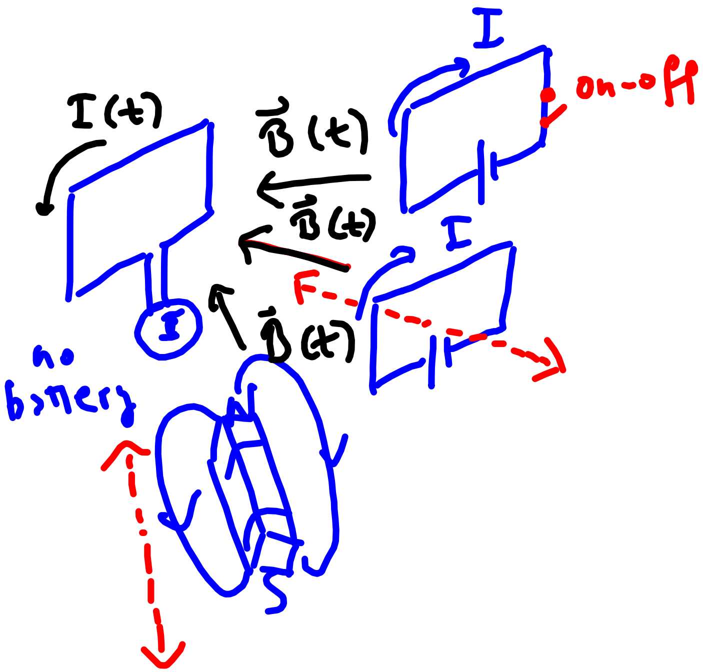
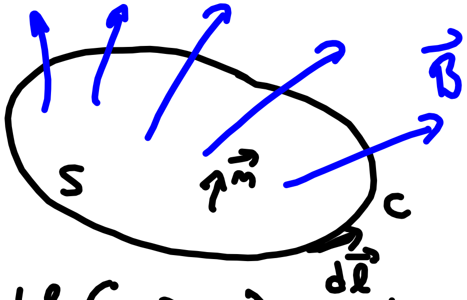
5.15 Faraday's law of induction

Third experimental law that we will introduce.





Flux of mag. field.

$$F = \int_S \vec{B}\cdot\vec{n}\, da$$

$$\mathcal{E} \stackrel{def}{=} \oint_c \vec{E}\cdot d\vec{l}$$ electromotive force

$$\boxed{\mathcal{E} = -\frac{dF}{dt}}$$

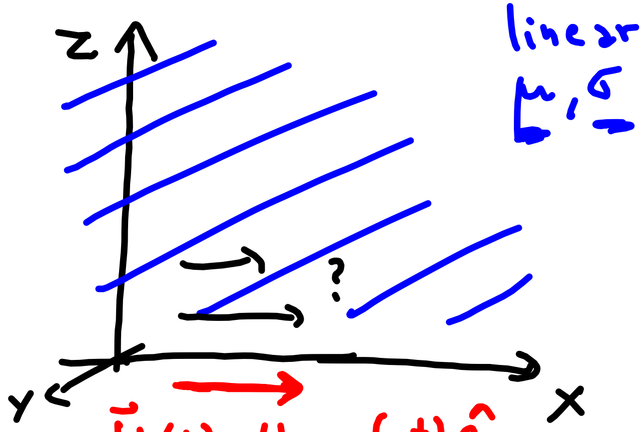
$$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{l} \underset{\substack{\uparrow \\ \text{math} \\ \text{Stokes's} \\ \text{theorem}}}{=} \int_S (\nabla\times\vec{E})\cdot\vec{n}\,da \underset{\substack{\uparrow \\ \text{Faraday's} \\ \text{law}}}{=} -\frac{d}{dt}\int_{\underset{\substack{\uparrow \\ \text{time independent}}}{S}} \vec{B}\cdot\vec{n}\,da = \int_S \left(-\frac{d\vec{B}}{dt}\right)\cdot\vec{n}\,da$$

$$\boxed{\nabla\times\vec{E}(\vec{x},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{x},t)}$$

generalization of $\nabla\times\vec{E} = 0$ (Ch 1–4).

5.18 Quasi-static magnetic fields

$\frac{\partial \vec{B}}{\partial t}$ induces $\vec{E}$, but if $\vec{B}(t)$ "slow", then $\frac{\partial \vec{B}}{\partial t}$ small, and $\vec{E}$ small.



$\nabla \cdot \vec{B} = 0$, $\quad \nabla \times \vec{H} = \vec{J} \underbrace{+ \cdots}_{\text{small}}$

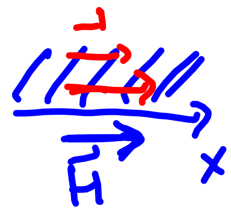
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\quad \vec{J} = \sigma \vec{E}$ ← not fundamental

$\nabla \times \vec{H} = \vec{J}$, linear $\nabla \times \vec{B} = \mu \vec{J}$

$$\nabla \times \vec{B} = \mu\sigma \vec{E}$$
$$\nabla \times (\nabla \times \vec{A}) = \mu\sigma \vec{E}$$
$$\nabla(\underbrace{\nabla \cdot \vec{A}}_{=0 \text{ Coul. gauge}}) - \nabla^2 \vec{A} = \mu\sigma \vec{E}$$
$$\nabla^2 \vec{A} = -\mu\sigma \vec{E}$$

$$\nabla \times \vec{E} = -\partial \vec{B}/\partial t = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$\boxed{\nabla^2 \vec{A} = \mu\sigma \frac{\partial \vec{A}}{\partial t}}$$



B.C. ① $(\vec{B}_2-\vec{B}_1)\cdot\vec{n}_{21}=0 \longleftarrow \nabla\cdot\vec{B}=0$

② $(\vec{H}_2-\vec{H}_1)\times\vec{n}_{21}=0 \longleftarrow \nabla\times\vec{H}=\vec{J}$

Try $\vec{A}=(0,0,\mu y H(z,t))$

$$\vec{H}=\frac{\vec{B}}{\mu}=\frac{\nabla\times\vec{A}}{\mu}=\frac{1}{\mu}\begin{vmatrix}\hat{e}_x & \hat{e}_y & \hat{e}_z\\ \nabla_x & \nabla_y & \nabla_z\\ 0 & 0 & \mu y H(z,t)\end{vmatrix}=\hat{e}_x H(z,t)$$

✓ if $H(0,t)=H_0\cos(\omega t)$.

$$\nabla^2 A_z=\nabla^2[\mu y H(z,t)]=\mu y\frac{\partial^2}{\partial z^2}H(z,t)=\mu\sigma\underbrace{\mu y\frac{\partial H(z,t)}{\partial t}}_{\frac{\partial \vec{A}}{\partial t}}$$

$\Downarrow$

$$\boxed{\frac{\partial^2}{\partial z^2}H(z,t)=\mu\sigma\frac{\partial H(z,t)}{\partial t}}$$

$$H(z,t) = h(z)\, e^{-i\omega t}$$

$$\frac{\partial^2 H}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left(h\, e^{-i\omega t}\right) = \cancel{e^{-i\omega t}}\, \frac{\partial^2 h}{\partial z^2}$$

equal

$$\mu\sigma \frac{\partial}{\partial t}\left(h\, e^{-i\omega t}\right) = \mu\sigma\, h(z)(-i\omega)\, \cancel{e^{-i\omega t}}$$

$$\frac{\partial^2 h(z)}{\partial z^2} = -i\omega\mu\sigma\, h(z) \quad ; \quad h(z) = e^{ikz}$$

complex

$$-k^2 e^{ikz} = -i\omega\mu\sigma\, e^{ikz} \quad ; \quad k = \pm\sqrt{i\omega\mu\sigma}$$

$$\sqrt{i} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

$$h(z) = e^{\pm i(1+i)\sqrt{\frac{\omega\mu\sigma}{2}}\, z}$$

$$= e^{\pm(i-1)\sqrt{\cdots}\, z}$$

$$\boxed{\begin{gathered} H(z,t) = A\, e^{-\frac{z}{\delta}}\, e^{i\left(\frac{z}{\delta} - \omega t\right)} \\ \vec{H} = H(z,t)\, \hat{e}_x \end{gathered}}$$

$$\boxed{\delta = \sqrt{\frac{2}{\mu\sigma\omega}}} \quad \text{length "skin depth"}$$

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

→ if $\sigma = \infty$, $\delta = 0$

→ if $\omega = 0$, $\delta \to \infty$, $\vec{H}$ fields constant for all z

Electric field ?

$$\nabla \times \vec{H} = \vec{J} = \sigma \vec{E}$$

$\vec{A} = (0, 0, A_z)$

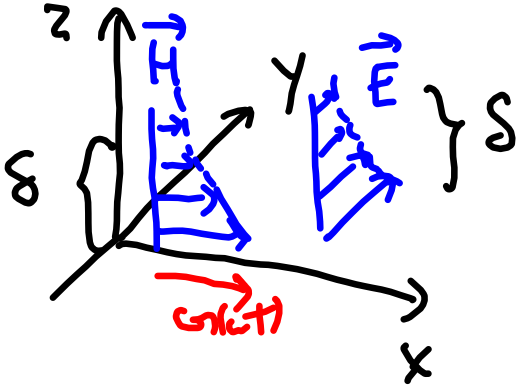
$\vec{H} = (H_x, 0, 0)$

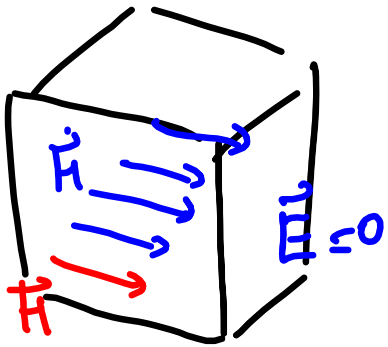
$\vec{E} = (0, E_y, 0)$

$$\boxed{E_y = A\frac{\mu\omega}{\sqrt{2}}\,\delta\, e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t + \frac{3\pi}{4}\right)}$$

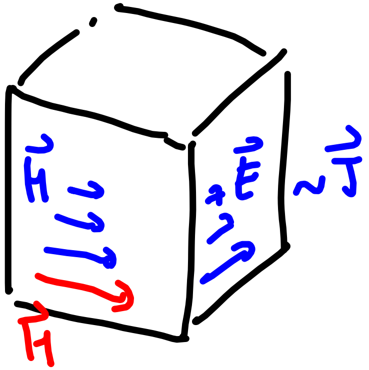
$$E_y \xrightarrow[\omega \to 0]{} 0$$

$\vec{J} = \sigma \vec{E}$

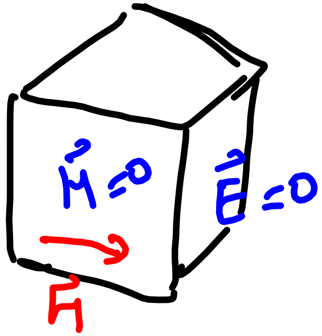




$\omega = 0$
$\delta \to \infty$



$\omega = \text{finite}$



$\omega \to \infty$
$\delta \to 0$