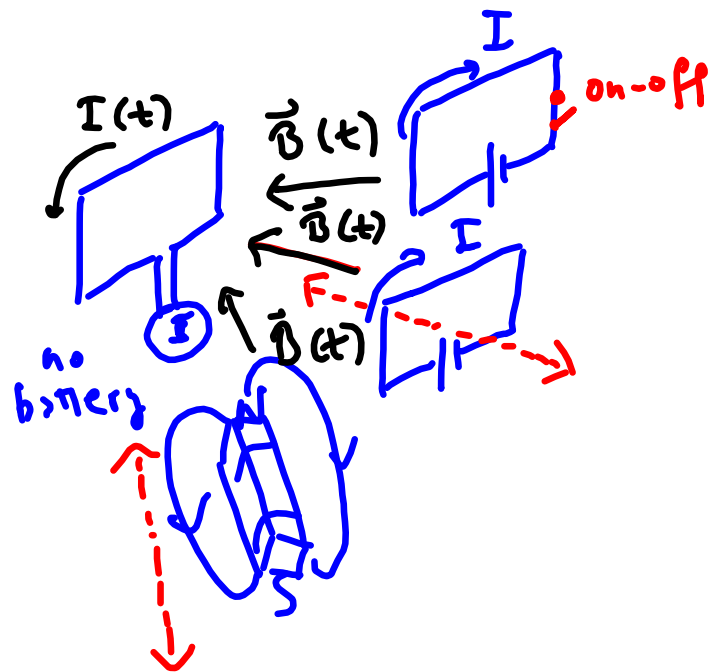


5.15 Faraday's law of induction

Third experimental law that we will introduce.



The diagram shows a surface S bounded by a curve C . A magnetic field \vec{B} is shown passing through the surface. The flux of the magnetic field is given by $F = \int_S \vec{B} \cdot \vec{n} da$. The electromotive force is given by $\epsilon \stackrel{\text{def}}{=} \int_C \vec{E} \cdot d\vec{l}$.

$$F = \int_S \vec{B} \cdot \vec{n} da$$

Flux of mag. field.

$$\epsilon \stackrel{\text{def}}{=} \int_C \vec{E} \cdot d\vec{l}$$

electromotive force

$$\epsilon = - \frac{dF}{dt}$$

$$\mathcal{E} = \oint_C \vec{E} \cdot d\vec{\ell} \stackrel{\substack{\uparrow \\ \text{math } S \\ \text{Stoke's} \\ \text{theorem}}}{=} \int_S (\nabla \times \vec{E}) \cdot \vec{n} da \stackrel{\substack{\uparrow \\ \text{Faraday's} \\ \text{law}}}{=} - \frac{d}{dt} \int_S \vec{B} \cdot \vec{n} da = \int_S \left(- \frac{d\vec{B}}{dt} \right) \cdot \vec{n} da$$

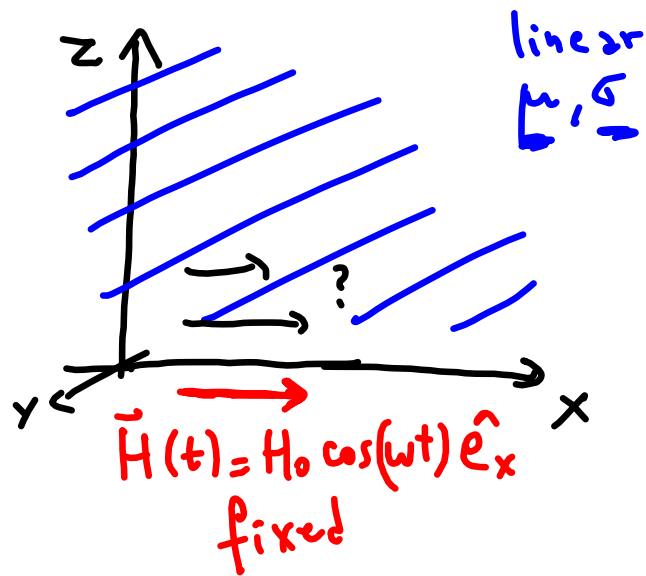
time independent

$$\nabla \times \vec{E}(\vec{x}, t) = - \frac{\partial \vec{B}(\vec{x}, t)}{\partial t}$$

Generalization of $\nabla \times \vec{E} = 0$ (Ch 1-4).

5.18 Quasi-static magnetic fields

$\frac{\partial \vec{B}}{\partial t}$ induces \vec{E} , but if $\vec{B}(t)$ "slow", then $\frac{\partial \vec{B}}{\partial t}$ small, and \vec{E} small.



$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{J} + \dots$$

small

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{J} = \sigma \vec{E}$$

↑ not fundamental

$$\nabla \times \vec{H} = \vec{J}, \text{ linear} \quad \nabla \times \vec{B} = \mu \vec{J}$$

$$\left. \begin{aligned} \nabla \times \vec{B} &= \mu \sigma \vec{E} \\ \nabla \times (\nabla \times \vec{A}) &= \mu \sigma \vec{E} \\ \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} &= \mu \sigma \vec{E} \end{aligned} \right\} \begin{aligned} \nabla \times \vec{E} &= -\partial \vec{B} / \partial t \\ &= \nabla \times (-\partial \vec{A} / \partial t) \end{aligned}$$

= Coul. gauge

$$\nabla^2 \vec{A} = -\mu \sigma \vec{E}$$

$$\nabla^2 \vec{A} = \mu \sigma \frac{\partial \vec{A}}{\partial t}$$

B.C. ① $(\vec{B}_2 - \vec{B}_1) \cdot \vec{n}_{21} = 0 \leftarrow \nabla \cdot \vec{B} = 0$

② $(\vec{H}_2 - \vec{H}_1) \times \vec{n}_{21} = 0 \leftarrow \nabla \times \vec{H} = \vec{J}$



Try $\vec{A} = (0, 0, \mu y H(z,t))$

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{\nabla \times \vec{A}}{\mu} = \frac{1}{\mu} \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \nabla_x & \nabla_y & \nabla_z \\ 0 & 0 & \mu y H(z,t) \end{vmatrix} = \hat{e}_x H(z,t)$$

✓ if $H(0,t) = H_0 \cos(\omega t)$

$$\nabla^2 A_z = \nabla^2 (\mu y H(z,t)) = \mu y \frac{\partial^2 H(z,t)}{\partial z^2} = \mu \sigma \underbrace{\mu y \frac{\partial H(z,t)}{\partial t}}_{\frac{\partial \vec{A}}{\partial t}}$$

$$\frac{\partial^2 H(z,t)}{\partial z^2} = \mu \sigma \frac{\partial H(z,t)}{\partial t}$$

$$H(z, t) = h(z) e^{-i\omega t}$$

$$\frac{\partial^2 H}{\partial z^2} = \frac{\partial^2 (h e^{-i\omega t})}{\partial z^2} = \cancel{e^{-i\omega t}} \frac{\partial^2 h}{\partial z^2} \quad \leftarrow \text{equal}$$

$$\mu\sigma \frac{\partial (h e^{-i\omega t})}{\partial t} = \mu\sigma h(z) (-i\omega) \cancel{e^{-i\omega t}}$$

$$\frac{\partial^2 h(z)}{\partial z^2} = -i\omega\mu\sigma h(z)$$

$$-k^2 e^{ikz} = -i\omega\mu\sigma e^{ikz}$$

$$h(z) = e^{ikz} \quad \uparrow \text{complex}$$

$$k = \pm \sqrt{i\omega\mu\sigma}$$

$$\sqrt{i} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

$$h(z) = e^{\pm i(1+i)\sqrt{\frac{\omega\mu\sigma}{2}} z}$$

$$= e^{\pm(i-i)\sqrt{\dots} z}$$

$$H(z, t) = A e^{-\frac{z}{\delta}} e^{i(\frac{z}{\delta} - \omega t)}$$

$$\vec{H} = H(z, t) \hat{e}_x$$

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

length "skin depth"

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \rightarrow \begin{cases} \text{if } \sigma = \infty, \delta = 0 \\ \text{if } \omega = 0, \delta \rightarrow \infty, \vec{H} \text{ fields constant for all } z \end{cases}$$

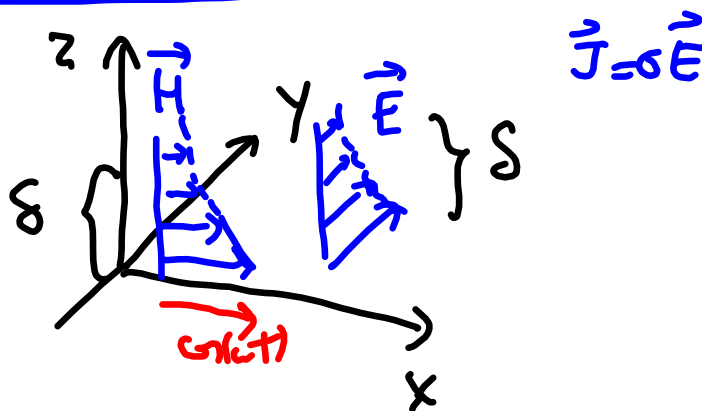
Electric field ?

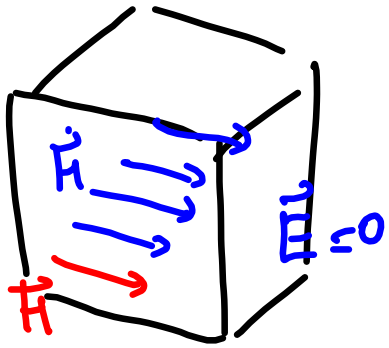
$$\nabla \times \vec{H} = \vec{J} = \sigma \vec{E}$$

$$\begin{aligned} \vec{A} &= (0, 0, A_z) \\ \vec{H} &= (H_x, 0, 0) \\ \vec{E} &= (0, E_y, 0) \end{aligned}$$

$$E_y = \frac{\mu\omega}{\sqrt{2}} \delta e^{-z/\delta} \cos\left(\frac{z}{\delta} - \omega t + \frac{3\pi}{4}\right)$$

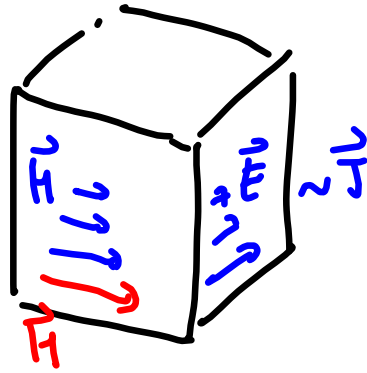
$$\begin{aligned} E_y &\rightarrow 0 \\ \omega &\rightarrow 0 \end{aligned}$$



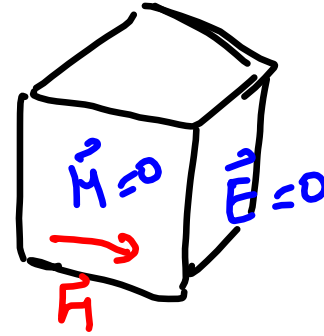


$$\omega = 0$$

$$\delta \rightarrow \infty$$



$$\omega = \text{finite}$$



$$\omega \rightarrow \infty$$

$$\delta \rightarrow 0$$