6.1 Maxwell Equations

Thus far: $\nabla\cdot\vec{B}=0$ $\quad\nabla\times\vec{H}=\vec{J}$ ← $\vec{J}_{eff}$

$\nabla\cdot\vec{D}=\rho$ $\quad\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

$\nabla\cdot\vec{J}=0$ must be replaced by $\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$ continuity Eq.
ch.5

$$\nabla\cdot\vec{J}+\frac{\partial(\nabla\cdot\vec{D})}{\partial t}=0$$

$$\nabla\cdot\Big(\underbrace{\vec{J}+\frac{\partial\vec{D}}{\partial t}}_{\vec{J}_{eff}}\Big)=0$$

$$\boxed{\begin{array}{ll}\nabla\cdot\vec{B}=0\ ; & \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}\\ \nabla\cdot\vec{D}=\rho\ ; & \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\end{array}}$$

$\frac{\partial\vec{D}}{\partial t}$ source of $\vec{H}$

$\frac{\partial\vec{B}}{\partial t}$ source of $\vec{E}$

New Eq. leads to "radiation"

6.2 Vector and Scalar potentials

$$\nabla\cdot\vec{B}=0 \Rightarrow \vec{B}=\nabla\times\vec{A}$$

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0\ ,\quad \nabla\times\vec{E}+\frac{\partial}{\partial t}(\nabla\times\vec{A})=0\ ,\quad \nabla\times\Big(\underbrace{\vec{E}+\frac{\partial\vec{A}}{\partial t}}_{-\nabla\phi}\Big)=0$$

$$\boxed{\vec{E}=-\nabla\phi-\frac{\partial\vec{A}}{\partial t}}$$

$$\nabla\cdot\vec{D}=\rho\ ;\ \stackrel{\text{vacuum}}{\vec{D}=\epsilon_0\vec{E}} \qquad \epsilon_0\nabla\cdot\Big(-\nabla\phi-\frac{\partial\vec{A}}{\partial t}\Big)=\rho$$

$$\Downarrow$$

I $$\boxed{\nabla^2\phi+\underbrace{\frac{\partial}{\partial t}(\nabla\cdot\vec{A})}_{\text{New Term}}=-\frac{\rho}{\epsilon_0}}$$

Replaces the Poisson Eq.

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

vacuum

$$\underbrace{\frac{1}{\mu_0}\left(\nabla \times \vec{B}\right)}_{\frac{1}{\mu_0}\nabla\times(\nabla\times\vec{A})} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J} + \epsilon_0 \frac{\partial}{\partial t}\overbrace{\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right)}^{\vec{E}}$$

$$\underbrace{\frac{1}{\mu_0}\nabla\times(\nabla\times\vec{A})}_{\frac{1}{\mu_0}\left(\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}\right)} \qquad = \vec{J} - \nabla\left(\epsilon_0 \frac{\partial\phi}{\partial t}\right) - \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}.$$

$$\text{II} \quad \boxed{\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial\phi}{\partial t}\right) = -\mu_0 \vec{J}}$$

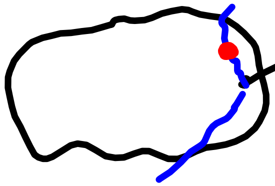
$$\mu_0\epsilon_0 = \frac{1}{c^2} \qquad 1/c^2$$

Gauge invariance

$\vec{B} = \nabla \times \vec{A} \longrightarrow \vec{A}' = \vec{A} + \nabla\Lambda \leftarrow$ scalar   same $\vec{B}$

$\vec{E} = -\nabla\phi - \dfrac{\partial \vec{A}}{\partial t}$ $\rightarrow$ $\vec{A}' = \vec{A} + \nabla\Lambda$, $\quad \phi' = \phi - \dfrac{\partial \Lambda}{\partial t}$ same $\vec{E}$

$\Lambda = \Lambda(\vec{x}, t)$



$\{\vec{A}\}$

$$\boxed{\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0}$$

Lorentz gauge

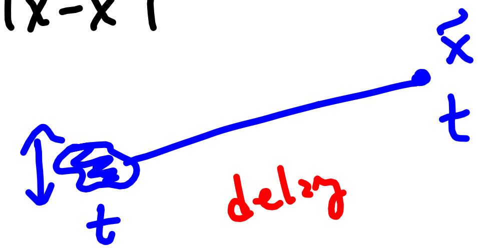
II $\rightarrow$ $$\boxed{\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}}$$

I $\rightarrow$ $$\boxed{\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}}$$

Another gauge: $\nabla \cdot \vec{A} = 0$ Coulomb gauge

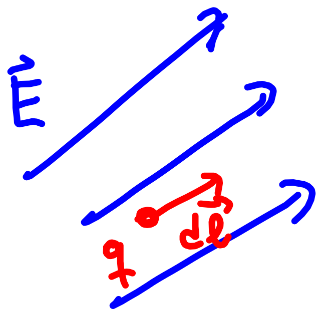
(I) $$\nabla^2 \phi(\vec{x},t) = -\frac{\rho(\vec{x},t)}{\epsilon_0}$$ same as Poisson Eq. but with t-dependence

$$\phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}',t)\, d^3x'}{|\vec{x}-\vec{x}'|}$$ Instantaneous



$$\vec{E} = -\underbrace{\nabla\phi}_{\text{Inst.}} - \underbrace{\frac{\partial \vec{A}}{\partial t}}_{\text{not Inst.}}$$

6.7 Conservation of energy



$$\underbrace{\frac{dW}{dt}}_{\substack{\text{Power on} \\ \text{Rate of} \\ \text{doing work}}} = \frac{\vec{F}\cdot d\vec{\ell}}{dt} = q\frac{\vec{E}\cdot d\vec{\ell}}{dt} = q\,\vec{E}\cdot\vec{v}$$

$$\downarrow$$

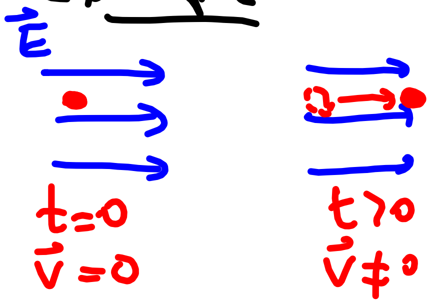
$$\sum_{i=1}^{N} q_i \vec{v}_i \cdot \vec{E}_i$$

$$\downarrow \text{ continuum}$$

$$\int_V \vec{J}(\vec{x},t)\,\vec{E}(\vec{x},t)\,d^3x$$

$$\boxed{\frac{dW}{dt} = \int_V \vec{J}\cdot\vec{E}\,d^3x}$$

Example:



$$\vec{J}\cdot\vec{E} \longrightarrow q\vec{v}\cdot\vec{E} > 0$$

$$\frac{dW}{dt} > 0$$

$$\frac{dE_{\underbrace{\text{particle}}_{\text{mechanical}}}}{dt} > 0$$

$$\int \vec{J}\cdot\vec{E}\, d^3x = \int d^3x\, \vec{E}\cdot\underbrace{\left(\nabla\times\vec{H} - \frac{\partial \vec{D}}{\partial t}\right)}_{\vec{J}\text{ (Max. Eqs)}} =$$

$$\vec{E}\cdot(\nabla\times\vec{H}) = -\nabla\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot(\nabla\times\vec{E})$$

$$= \int d^3x \left[-\nabla\cdot(\vec{E}\times\vec{H}) + \vec{H}\cdot\underbrace{(\nabla\times\vec{E})}_{-\frac{\partial \vec{B}}{\partial t}\text{ (Max. Eqs)}} - \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right]$$

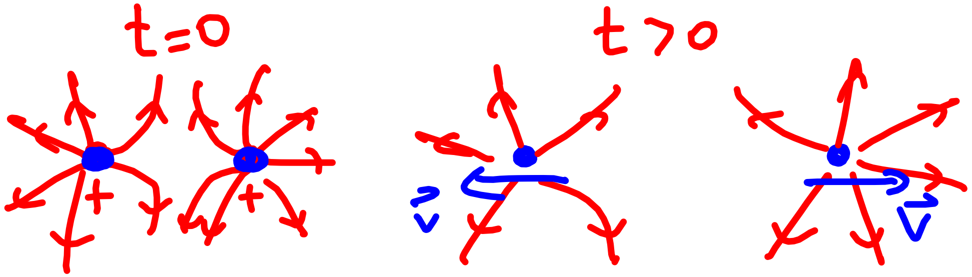
linear

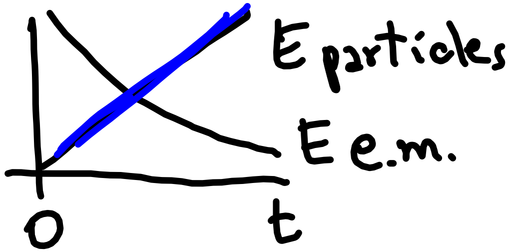
$$\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} = \epsilon\left(\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}\right) = \frac{\epsilon}{2}\frac{\partial(\vec{E}\cdot\vec{E})}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) \;;\quad \vec{H}\cdot\frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\frac{\partial(\vec{H}\cdot\vec{B})}{\partial t}$$

$$\boxed{\int \vec{J}\cdot\vec{E}\, d^3x = -\int d^3x\, \nabla\cdot(\vec{E}\times\vec{H}) - \frac{\partial}{\partial t}\int d^3x \underbrace{\left(\frac{\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}}{2}\right)}_{\text{energy density } u}}$$

$$\boxed{\vec{J}\cdot\vec{E} = -\nabla\cdot\underbrace{(\vec{E}\times\vec{H})}_{\text{Poynting vector } \vec{S}=\vec{E}\times\vec{H}} - \frac{\partial u}{\partial t}}$$

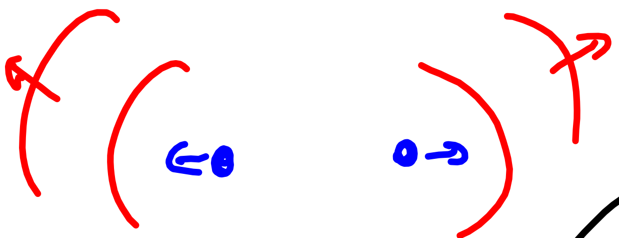
Example





$$\underbrace{\int \vec{J}\cdot\vec{E}\, d^3x}_{\substack{>0 \\ \frac{dE_{mech}}{dt}}} + \underbrace{\frac{\partial}{\partial t}\int \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\, d^3x}_{\substack{<0 \\ \frac{dE_{em}}{dt}}} = -\int d^3x\, \nabla\cdot\vec{S}$$

$$= -\oint \underbrace{\vec{S}\cdot\vec{n}\, ds}_{\substack{S \\ \neq 0}}$$



Class. Mech. analog