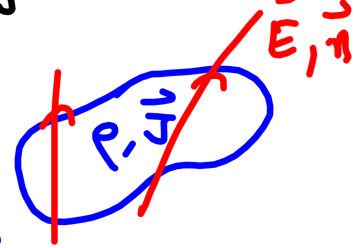
# Conservation of momentum

For one particle $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \longrightarrow \sum_i^N q_i(\vec{E}_i + \vec{v}_i \times \vec{B}_i)$

Continuum is reached by $q \to \rho$ ; $q\vec{v} \longrightarrow \vec{J}$




$$\vec{F} = \int d^3x \left[\underline{\rho(\vec{x})\vec{E}(\vec{x})} + \underline{\vec{J}(\vec{x})\times\vec{B}(\vec{x})}\right]$$

$\nabla\cdot\vec{D} = \rho$

vacuum

$\epsilon_0 \nabla\cdot\vec{E} = \rho$

$\epsilon_0(\nabla\cdot\vec{E})\vec{E} = \rho\vec{E}$

$\underset{\uparrow\ \frac{1}{\mu_0\epsilon_0}}{c^2}(\nabla\cdot\vec{B})\vec{B} = 0$ M.Eq.

$\underset{\text{M.Eq.}}{\vec{J} = \nabla\times\vec{H}} - \frac{\partial\vec{D}}{\partial t} \overset{\text{vacuum}}{\Leftarrow} \frac{\nabla\times\vec{B}}{\mu_0} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\vec{J}\times\vec{B} = \frac{(\nabla\times\vec{B})\times\vec{B}}{\mu_0} \underbrace{- \epsilon_0 \frac{\partial\vec{E}}{\partial t}\times\vec{B}}$$

$$\epsilon_0\frac{\partial(\vec{B}\times\vec{E})}{\partial t} = \epsilon_0 \underbrace{\frac{\partial\vec{B}}{\partial t}}_{\text{M.Eq. } -\nabla\times\vec{E}}\times\vec{E} + \epsilon_0\,\vec{B}\times\frac{\partial\vec{E}}{\partial t}$$

$$\rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}(\nabla\cdot\vec{E})+c^2\vec{B}(\nabla\cdot\vec{B})-\vec{E}\times(\nabla\times\vec{E})-\underset{\frac{1}{\mu_0\epsilon_0}}{c^2}\vec{B}\times(\nabla\times\vec{B})\right]$$

$$-\epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\left.\vec{E}(\nabla\cdot\vec{E})-\vec{E}\times(\nabla\times\vec{E})\right|_{\alpha=1,2,3} \overset{HW}{=} \sum_{\substack{\beta\\1,2,3}}\frac{\partial}{\partial x_\beta}\left[E_\alpha E_\beta-\frac{1}{2}(\vec{E}\cdot\vec{E})\delta_{\alpha\beta}\right]$$

Maxwell's stress tensor

$$\boxed{T_{\alpha\beta} \overset{def}{=} \epsilon_0\left[E_\alpha E_\beta+c^2B_\alpha B_\beta-\frac{1}{2}(\vec{E}\cdot\vec{E}+c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\right]}$$

$$\underbrace{\int_V (\rho\vec{E} + \vec{J}\times\vec{B})\Big|_\alpha d^3x}_{\vec{F}|_\alpha = \frac{d}{dt}\vec{P}_{mech}} + \underbrace{\epsilon_0 \int_V \frac{\partial}{\partial t}(\vec{E}\times\vec{B})\Big|_\alpha d^3x}_{\frac{d}{dt}\vec{P}_{em}} = \underbrace{\int \sum_\beta \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x}$$

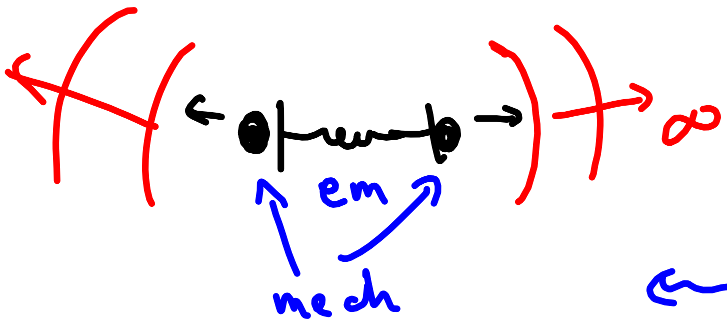
$$\vec{P}_{em} = \frac{1}{c^2}\int d^3x\, (\underbrace{\vec{E}\times\vec{H}}_{\vec{S}})$$

$$\nabla\cdot\vec{a} = \sum_\beta \frac{\partial a_\beta}{\partial x_\beta}$$

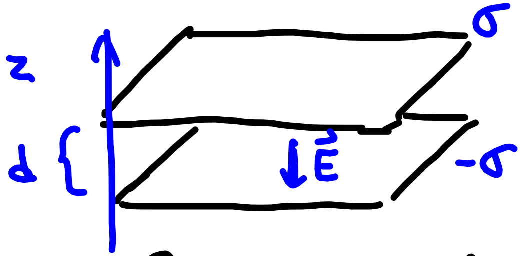
$$\Downarrow$$

$$\oint_S dS\, \underbrace{\sum_\beta T_{\alpha\beta}\, n_\beta}_{\vec{T}^\alpha\cdot\vec{n}}$$

Flow of momentum across surface S




## Example 1




$$|\vec{E}| = \frac{\sigma}{\epsilon_0}$$

Force on upper plane?

$$E_x = E_y = 0, \quad E_z = -\frac{\sigma}{\epsilon_0}$$

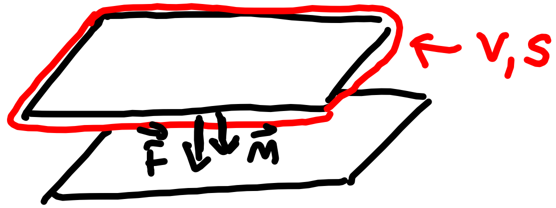
$$B_x = B_y = B_z = 0$$

$$T_{\alpha\beta} = \epsilon_0 \left[ E_\alpha E_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E})\delta_{\alpha\beta} \right]$$

$$T_{11} = \epsilon_0 \left[ 0 - \frac{1}{2}(E_z)^2 \right] = -\frac{\sigma^2}{2\epsilon_0}$$

$$T_{22} = -\frac{\sigma^2}{2\epsilon_0}$$

$$T_{33} = +\frac{\sigma^2}{2\epsilon_0} \qquad T = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix} \frac{\sigma^2}{2\epsilon_0}$$

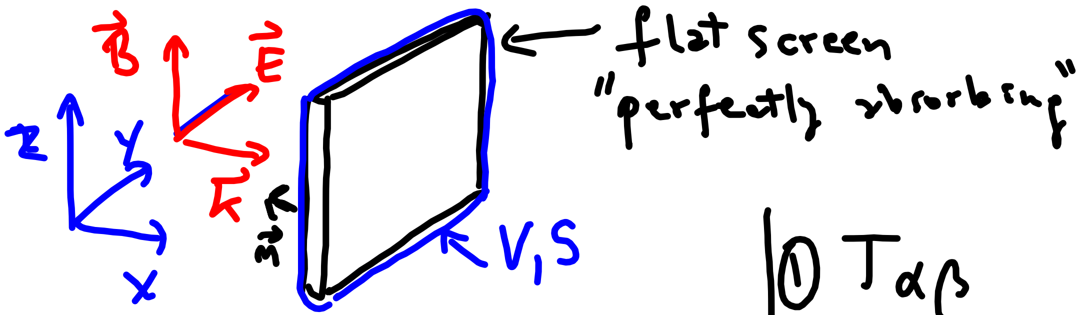



$$\oint_S \sum_\beta T_{3\beta}\, \underbrace{n_\beta}_{(0,0,-1)}\, dS$$

$$= T_{33}(-1)\,\text{Area}$$

$$= -\frac{\sigma^2}{2\epsilon_0}\,\text{Area} = \text{Force}$$

# Example 2 (Problem 11 of Chapter 6)




$$\vec{E} = E_0 \cos(kx - \omega t)\hat{j}$$

$$\vec{B} = \frac{E_0}{c} \cos(kx - \omega t)\hat{k}$$

① $T_{\alpha\beta}$

$E_1 = 0,\ E_2 \neq 0,\ E_3 = 0$

$B_1 = 0,\ B_2 = 0,\ B_3 \neq 0$

$E_\alpha E_\beta,\ B_\alpha B_\beta$

- $T_{\alpha\beta} = 0$ if $\alpha \neq \beta$
- $T_{\alpha\alpha} \neq 0,\ \alpha = 1, 2, 3$

$$T_{11} = \epsilon_0 \Big[ \underset{=0}{E_1^2} + c^2 \underset{=0}{B_1^2} - \frac{1}{2}\Big[ \underbrace{(\vec{E}\cdot\vec{E})}_{E_0^2 \cos^2} + \cancel{c^2} \underbrace{(\vec{B}\cdot\vec{B})}_{\frac{E_0^2}{\cancel{c^2}}\cos^2} \Big] \underbrace{\delta_{11}}_{1} \Big] = -\epsilon_0 E_0^2 \cos^2$$

$$T_{22} = \epsilon_0 \left( \frac{E_0^2}{2} - \frac{c^2 B_0^2}{2} \right) = 0 \qquad B_0 = \frac{E_0}{c}$$

$$T_{33} = 0$$

$$T = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

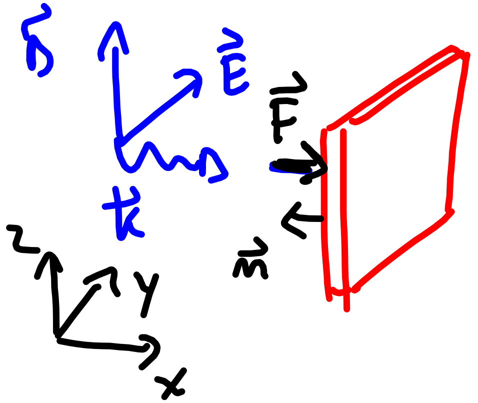
$$\sum_\beta T_{\alpha\beta} n_\beta = -T_{11} = +\epsilon_0 E_0^2 \cos^2(\omega t) \qquad n_\beta \to (-1,0,0)$$

$$= +\frac{\epsilon_0 E_0^2}{2}$$ time average

Time average $\frac{1}{T}\int_0^T dt \ldots \qquad T = \frac{2\pi}{\omega}$

$$\frac{1}{T}\int_0^T dt \cos^2(\omega t) = \frac{1}{2}$$

Force along the x axis $= \oint_S \sum_\beta T_{1\beta} n_\beta \, da = \frac{\epsilon_0 E_0^2}{2} \cdot \text{Area}$




$$\boxed{\frac{\text{Force}}{\text{Area}} = \text{radiation pressure} = \frac{\epsilon_0 E_0^2}{2}}$$

time average

energy density

$u \sim \vec{E}^2 + c^2 \vec{B}^2$

$\vec{P}_{\text{plane wave}}$ per unit volume $= \frac{\epsilon_0}{V} \int_V (\vec{E} \times \vec{B}) \, d^3x \doteq \hat{e}_x \frac{\epsilon_0 E_0^2}{2c} = \underbrace{\frac{u}{c}}_{} \hat{e}_x$

$\frac{E}{c} = p$