Ch. 7

7.1 Plane waves in a non-conducting medium

$\rho = 0$, $\vec{J} = 0$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0$$

Try a solution with a single frequency ω

$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x})\, e^{-i\omega t} \qquad \vec{B}(\vec{x},t) = \vec{B}(\vec{x})\, e^{-i\omega t}$$

$$\vec{D}(\vec{x},t) = \vec{D}(\vec{x})\, e^{-i\omega t}$$

$$\frac{\partial \vec{D}}{\partial t} = -i\omega \vec{D} \qquad \frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

$$\nabla\cdot\vec{B}=0\ ,\quad \nabla\times\vec{E}-i\omega\vec{B}=0 \qquad (1)$$

$$\nabla\cdot\vec{D}=0\ ,\quad \nabla\times\vec{H}+i\omega\vec{D}=0 \qquad (2) \quad \rightarrow \nabla\times\frac{\vec{B}}{\mu}+i\omega\epsilon\vec{E}=0$$

Assume linear materials $\vec{D}=\epsilon\vec{E}$, $\vec{B}=\mu\vec{H}$

(1) $\nabla\times\vec{E}-i\omega\vec{B}=0$

$$\underbrace{\nabla\cdot(\nabla\times\vec{E})}_{=0}-i\omega\nabla\cdot\vec{B}=0$$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times(\nabla\times\vec{E})-i\omega\underbrace{\nabla\times\vec{B}}_{-i\omega\mu\epsilon\vec{E}}=0$$

$$\nabla(\underbrace{\nabla\cdot\vec{E}}_{=0})-\nabla^2\vec{E}$$

$\nabla\cdot\vec{D}=0$

$$\boxed{\nabla^2\vec{E}+\mu\epsilon\omega^2\vec{E}=0}$$

only one ω
linear medium

From (2) $\nabla\times\vec{B} + i\omega\mu\epsilon\vec{E} = 0$

$\nabla\cdot\underbrace{(\nabla\times\vec{B})}_{=0} + i\omega\mu\epsilon\nabla\cdot\vec{E} = 0$

$\nabla\cdot\vec{E} = 0$ ✓

$\underbrace{\nabla\times(\nabla\times\vec{B})}_{\nabla(\underbrace{\nabla\cdot\vec{B}}_{=0}) - \nabla^2\vec{B}} + i\omega\mu\epsilon\underbrace{\nabla\times\vec{E}}_{i\omega\vec{B}} = 0$

$$\boxed{\nabla^2\vec{B} + \mu\epsilon\omega^2\vec{B} = 0}$$

$$\boxed{\nabla^2\vec{E} + \mu\epsilon\omega^2\vec{E} = 0}$$

Helmholtz wave equations

1D, Try $e^{\pm ikx}e^{-i\omega t}$

$\nabla^2 e^{\pm ikx} = -k^2 e^{\pm ikx}$

$$\boxed{-k^2 + \mu\epsilon\omega^2 = 0}$$

$$k = \sqrt{\mu\varepsilon}\,\omega$$

$$v \overset{\text{def}}{=} \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{\mu_0\epsilon_0}{\mu_0\epsilon_0}} = \underbrace{\sqrt{\frac{\mu_0\epsilon_0}{\mu\epsilon}}}_{1/n}\ \overbrace{\frac{1}{\sqrt{\mu_0\epsilon_0}}}^{c} = \boxed{\frac{c}{n} = v}$$

phase velocity

1/n ← index of diffraction

↳ Assume non-dispersive medium. In general, $\epsilon = \epsilon(k, \omega)$

$$u(x,t) = a\,e^{ikx}e^{-i\omega t} + b\,e^{-ikx}e^{-i\omega t} = a\,e^{ik(x-vt)} + b\,e^{-ik(x+\omega t)}$$

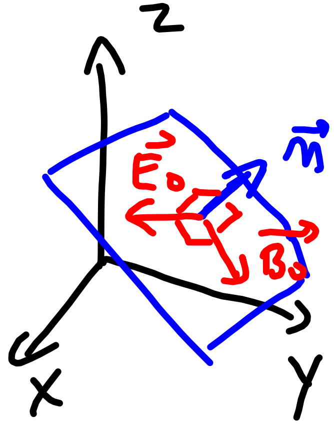
Extension to 3D

$\vec{k} = k\vec{n}$

Try $\vec{E}(\vec{x},t) = \vec{E}_0\, e^{ik\vec{n}\cdot\vec{x}}\, e^{-i\omega t}$ ↑ just 1 frequency

$\vec{B}(\vec{x},t) = \vec{B}_0\, e^{ik\vec{n}\cdot\vec{x}}\, e^{-i\omega t}$

$$\nabla^2 e^{ik\vec{n}\cdot\vec{x}} = \underbrace{(ik\vec{n})^2}_{-k^2 \underbrace{\vec{n}\vec{n}}_{=1} = -k^2} e^{ik\vec{n}\cdot\vec{x}}$$



$$-k^2 + \mu\varepsilon\omega^2 = 0$$

$$\boxed{k = \sqrt{\mu\varepsilon}\,\omega}$$

$\nabla\times\vec{E} = i\omega\vec{B}$

$ik\vec{n}\times\vec{E}_0 = i\omega\vec{B}_0$

$\underbrace{\frac{k}{\omega}}_{n/c}\, \vec{n}\times\vec{E}_0 = \vec{B}_0$

$$\boxed{n(\vec{n}\times\vec{E}_0) = c\vec{B}_0}$$

$\nabla\cdot\vec{B} = 0$

$\nabla\cdot\left(\vec{B}_0\, e^{ik\vec{n}\cdot\vec{x}}\right) = 0$

$ik\vec{n}\cdot\vec{B}_0\, e^{ik\vec{n}\cdot\vec{x}} = 0$

$$\boxed{\vec{n}\cdot\vec{B}_0 = 0}$$

$\nabla\cdot\vec{D} = 0$

$$\boxed{\vec{n}\cdot\vec{E}_0 = 0}$$

Detail Time-averages

$$f(\vec{x},t) = a\cos(\vec{k}\cdot\vec{x}-\omega t)$$
$$g(\vec{x},t) = b\cos(\vec{k}\cdot\vec{x}-\omega t)$$

$$\underset{\uparrow \text{ time averages}}{\langle fg\rangle} \stackrel{\text{def}}{=} \frac{1}{\underset{\uparrow\ \frac{2\pi}{\omega}}{T}}\int_0^T dt\, fg = \frac{1}{T}\int_0^T dt\, ab\,\underbrace{\cos^2(\vec{k}\cdot\vec{x}-\omega t)}_{\frac{1}{2}(1+\cos[2(\vec{k}\cdot\vec{x}-\omega t)])} = \frac{ab}{2}.$$

Complex notation: $f = a e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

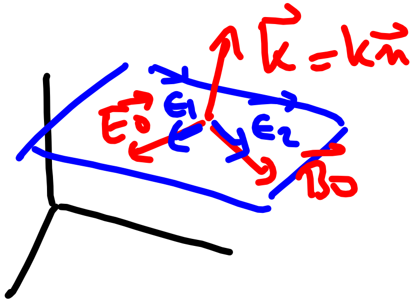
$g = b e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\langle fg\rangle = \frac{ab}{2} = \mathrm{Re}\left(\frac{fg^*}{2}\right)$$

$$\langle\vec{S}\rangle = \mathrm{Re}\left(\frac{\vec{E}\times\vec{H}^*}{2}\right)$$

$$\langle u\rangle = \frac{1}{4}\mathrm{Re}\left(\epsilon\vec{E}\cdot\vec{E}^* + \frac{1}{\mu}\vec{B}\cdot\vec{B}^*\right)$$

7.2 a. Linear polarization



$\vec{E}_0 \sim \vec{\epsilon}_1$ linear polarization

$\vec{B}_0 \sim \vec{\epsilon}_2$

b. circular polarization $\vec{E} = (\vec{\epsilon}_1 E_1 + \vec{\epsilon}_2 E_2) e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t}$

complex

Assume $\pi/2$ phase difference $e^{i\pi/2}$

$E_1 = E_0$, $E_2 = i E_0$

$\vec{E} = E_0 (\vec{\epsilon}_1 \pm i \vec{\epsilon}_2) e^{i\vec{k}\cdot\vec{x}} e^{-i\omega t}$

Example $\vec{k} = k\hat{e}_z$, $\vec{k}\cdot\vec{x} = kz$, $z=0$

$Re[(\vec{\epsilon}_1 + i\vec{\epsilon}_2) e^{-i\omega t}] = \vec{\epsilon}_1 \cos(\omega t) + \vec{\epsilon}_2 \sin(\omega t)$