Lorentz Transf. along the x_1 direction

$$x_0' = \gamma\,(x_0 - \beta x_1)$$
$$x_1' = \gamma\,(x_1 - \beta x_0)$$
$$x_2' = x_2$$
$$x'_3 = x_3$$

$x_0' = ct'$

$\beta = v/c$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$x_0'^2 - (x_1'^2 + x_2'^2 + x_3'^2) \overset{L.T.}{=}$$
$$= x_0^2 - (x_1^2 + x_2^2 + x_3^2)$$

invariant

$(c = c')$ Jackson

4-vectors :

In 3D, $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underset{3\times3}{\hat{A}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\begin{pmatrix} E_1' \\ E_2' \\ E_3' \end{pmatrix} = \underset{3\times3}{\hat{A}} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

$\vec{C}\cdot\vec{D}$ invariant

(A_0, A_1, A_2, A_3) is a 4-vector if it transforms like (x_0, x_1, x_2, x_3)

$$A_0' = \gamma\,(A_0 - \beta A_1)$$
$$A_1' = \gamma\,(A_1 - \beta A_0)$$
$$A_2' = A_2$$
$$A_3' = A_3$$

$$\boxed{C_0 D_0 - \vec{C}\cdot\vec{D}}$$
invariant

$$x_0' = \gamma(x_0 - \beta x_1) = \underbrace{\gamma}_{\frac{\partial x_0'}{\partial x_0}} x_0 - \underbrace{\gamma\beta}_{\frac{\partial x_0'}{\partial x_1}} x_1 \;; \qquad x'_\alpha = \sum_\beta \frac{\partial x'^\alpha}{\partial x_\beta} x_\beta$$

<u>Two types of 4-vectors:</u>

Contravariant → $A'^\alpha = \sum_\beta \dfrac{\partial x'^\alpha}{\partial x^\beta} A^\beta$

Covariant → $B'_\alpha = \sum_\beta \dfrac{\partial x^\beta}{\partial x'^\alpha} B_\beta$

$$F'^{\alpha\beta} = \sum_{\gamma,\delta} \frac{\partial x'^\alpha}{\partial x^{\underset{\uparrow\gamma}{\gamma}}} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$$

Tensor of rank two.

The combination $\sum_\alpha B_\alpha A^\alpha$ is invariant under L. Transf.

$B_\alpha A^\alpha$

Examples: $A^\alpha = (\Phi, \vec{A})$; $A_\alpha = (\Phi, -\vec{A})$

$\partial^\alpha = \left(\frac{\partial}{\partial x^0}, -\nabla\right)$; $\partial_\alpha = \left(\frac{\partial}{\partial x^0}, \nabla\right)$

Lorentz gauge: $\sum_\alpha \boxed{\partial_\alpha A^\alpha = 0} = \frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A}$

$g_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$; $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \underset{\text{D'alembertian}}{\Box} = \sum_\alpha \partial_\alpha \partial^\alpha$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J} \Longrightarrow \Box \vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho \Longrightarrow \Box \Phi = \frac{4\pi}{c}(c\rho)$$

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha \qquad J^\alpha = (\rho c, \vec{J})$$

$$\boxed{\Box A^\alpha = \frac{4\pi}{c} J^\alpha \, ; \quad \partial_\alpha A^\alpha = 0}$$

$$\boxed{F^{\alpha\beta} \overset{\text{def}}{=} \partial^\alpha A^\beta - \partial^\beta A^\alpha}$$

Now use $\vec{E}$ and $\vec{B}$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\Phi \longrightarrow E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$

$$\vec{B} = \nabla \times \vec{A} \longrightarrow B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$

$$F^{\alpha\beta} = \begin{array}{cccc|l} 0 & -E_x & -E_y & -E_z & \leftarrow \alpha=0 \\ E_x & 0 & -B_z & B_y & \leftarrow \alpha=1 \\ E_y & B_z & 0 & -B_x & \leftarrow \alpha=2 \\ E_z & -B_y & \boxed{B_x} & 0 & \leftarrow \alpha=3 \\ \uparrow \beta=0 & \uparrow \beta=1 & \uparrow \beta=2 & \uparrow \beta=3 & \end{array} \; ; \quad \mathfrak{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

Field Strength

F^{32}

Dual Field Strength

M. Eqs.

$$\boxed{\sum_\alpha \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

$\beta=0 \longrightarrow \nabla\cdot\vec{E} = 4\pi\rho$

$\beta=1,2,3 \longrightarrow \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

$$\boxed{\sum_\alpha \partial_\alpha \mathfrak{F}^{\alpha\beta} = 0}$$

$\beta=0 \longrightarrow \nabla\cdot\vec{B} = 0$

$\beta=1,2,3 \longrightarrow \nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

M. Eqs.

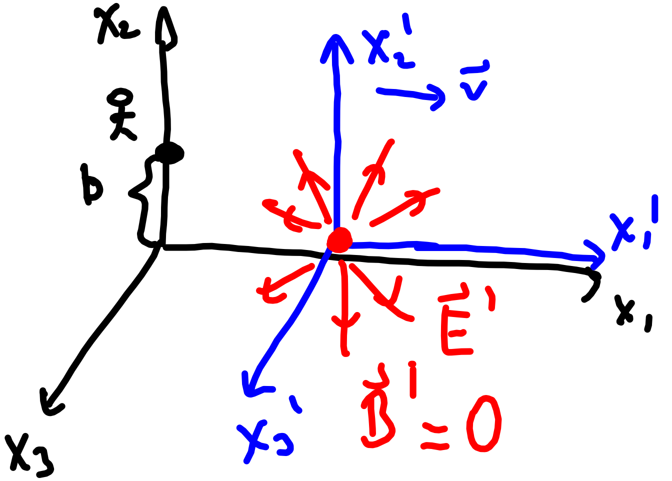
Since $\vec{E}$ and $\vec{B}$ are part of $F^{\alpha\beta}$, and we know how $F^{\alpha\beta}$ transforms, we know the new $\vec{E}'$ and $\vec{B}'$.

$$F'^{\alpha\beta} = \sum_{\gamma,\delta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$$

contains $\vec{E}'$ and $\vec{B}'$

contains $\vec{E}$ and $\vec{B}$

Sec. 11.10 Jackson



$E_1 = E_1'$ $\qquad B_1 = \cancel{B_1'}$

$E_2 = \gamma(E_2' + \beta \cancel{B_3'})$; $B_2 = \gamma(\cancel{B_2'} - \beta \cancel{E_3'}$

$E_3 = \gamma(\cancel{E_3'} - \beta \cancel{B_2'})$;

$B_3 = \gamma(\cancel{B_3'} + \beta E_2')$

$E_1 \neq 0 \quad B_1 = 0$

$E_2 \neq 0 \quad B_2 = 0$

$E_3 = 0 \quad B_3 \neq 0$