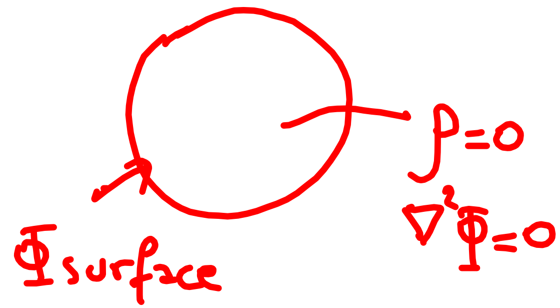
1.7 Poisson and Laplace Eqs.

$$= -\nabla\Phi \; ; \quad \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \; ; \quad \boxed{\nabla^2\Phi = -\frac{\rho(\vec{x})}{\epsilon_0}}$$

Poisson Eq.

If $\rho = 0$; $\boxed{\nabla^2\Phi = 0}$

Laplace Eq.



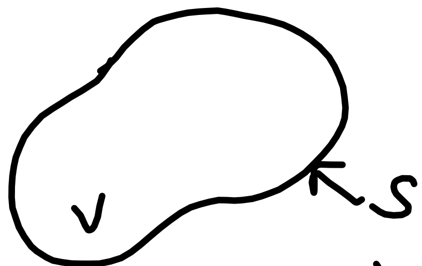
$$\nabla_x^2\Phi(x) = \frac{1}{4\pi\epsilon_0}\nabla_x^2\int\frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} = -\frac{\rho(\vec{x})}{\epsilon_0}$$

$$\boxed{\nabla_x^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')}$$

1.8 Green's theorem

$$\int_V \nabla\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\vec{n}\, da$$

Div. theorem



$\vec{A}$ "well behaved"

Special case

$$\vec{A} = \phi\nabla\psi$$

$$\nabla\cdot\vec{A} = \nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$$\vec{A}\cdot\vec{n} = \phi\,\boxed{\nabla\psi\cdot\vec{n}} = \phi\,\boxed{\frac{\partial\psi}{\partial n}}$$

notation

$$\int_V \left(\phi\nabla^2\psi + \underline{\nabla\phi\cdot\nabla\psi}\right)d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\,da$$

$$\vec{A} = \psi\nabla\phi$$

$$\int_V \left(\psi\nabla^2\phi\ \underline{----}\right) = \oint_S \psi\frac{\partial\phi}{\partial n}\,da$$

difference

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) d^3x' = \oint_S \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'}\right) da'$$

Green's theorem

Special case:

$$\phi = \Phi(\vec{x}') \; ; \; \psi = \frac{1}{|\vec{x} - \vec{x}'|}$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \; ; \; \nabla^2 \psi = \nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|}\right) = -4\pi\, \delta(\vec{x} - \vec{x}')$$

$$-4\pi \underbrace{\Phi(\vec{x})}_{\text{unknown}} + \int_V \frac{\rho(\vec{x}')\, d^3x'}{\epsilon_0 |\vec{x} - \vec{x}'|} = \oint_S \left(\Phi \frac{\partial}{\partial n'}\left(\frac{1}{|\vec{x} - \vec{x}'|}\right) - \frac{1}{|\vec{x} - \vec{x}'|} \underbrace{\frac{\partial \Phi}{\partial n'}}_{\vec{E}\cdot\vec{n}}\right) da'$$

Preliminary intro. to Green's functions

$$\hat{L}_x u(x) = f(x)$$

Ex. $\nabla^2 \Phi = -\frac{\rho(\vec{x})}{\epsilon_0}$

$$\hat{L}_x G(x,s) \stackrel{\text{def}}{=} \delta(x-s)$$

$$\hat{L}_x \int G(x,s) f(s)\, ds = \int \delta(x-s) f(s)\, ds = f(x) = \hat{L}_x u(x)$$

$$\boxed{u(x) = \int G(x,s) f(s)\, ds}$$

Depends on $\hat{L}$, geometry of the problem but does <u>not</u> depend on $f(x)$

1.10 Green functions and value problems

$$\nabla^2_{\vec{x}} G(\vec{x},\vec{x}') \stackrel{\text{def.}}{=} -4\pi\,\delta(\vec{x}-\vec{x}') \quad ; \quad G \stackrel{\text{special case}}{=} \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\nabla^2 G(\vec{x},\vec{x}') = \underbrace{\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|}}_{-4\pi\,\delta(\vec{x}-\vec{x}')} + \underbrace{\nabla^2 F(\vec{x},\vec{x}')}_{\nabla^2 F = 0}$$

Back to Green's functions, $\phi = \Phi$, $\psi = G$

$$\int_V \Big(\Phi \underbrace{\nabla^2 G}_{-4\pi\,\delta(\vec{x}-\vec{x}')} - G \underbrace{\nabla^2 \Phi}_{-\frac{\rho(\vec{x}')}{\epsilon_0}}\Big)\, d^3x' = \oint_S \Big(\Phi \frac{\partial G}{\partial n'} - G \frac{\partial \Phi}{\partial n'}\Big)\, da'$$

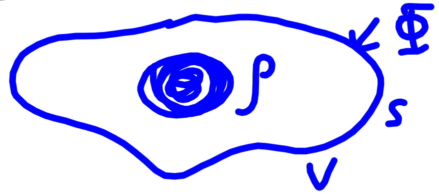
$$-4\pi\Phi(\vec{x}) + \frac{1}{\epsilon_0}\int \rho(\vec{x}')G(\vec{x},\vec{x}')\,d^3x' = \oint_S \left(\Phi\frac{\partial G}{\partial n'} - G\frac{\partial \Phi}{\partial n'}\right)da'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')\,d^3x' + \frac{1}{4\pi}\oint_S \left(\cancel{G\frac{\partial \Phi}{\partial n'}} - \Phi\frac{\partial G}{\partial n'}\right)da'$$

$$\boxed{\nabla^2 G_D = -4\pi\delta}\;; \quad G_D = 0 \text{ at the surface}$$

Dirichlet boundary condition

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}da'}$$



Steps

1. Find $G_D(\vec{x}, \vec{x}')$

2. Plug G_D, $\frac{\partial G_D}{\partial n}$, ρ, $\Phi_{surface}$ into previous Eq.

3. Integrate

$$G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})$$