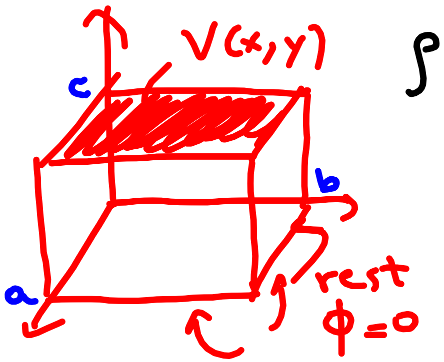


$\rho = 0$

$$\phi(x,y,z) = \sum_{n,m} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

$\alpha_n = \frac{\pi n}{a}$ $\quad$ $\beta_m = \frac{\pi m}{b}$ $\quad$ $\gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2}$

$$V(x,y) = \sum_{n,m} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c)$$

given $\quad$ unknown $\quad$ $z = c$

$$\frac{2}{a} \int_0^a dx \; \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m x}{a}\right) = \delta_{nm}$$

orthogonality relation

$$\int_0^a\!\!\int_0^b dx\,dy\,\underbrace{\sin(\alpha_\ell x)\sin(\beta_q y)}\;V(x,y) = \int_0^a\!\!\int_0^b \sum_{n,m} A_{nm}\,\underbrace{\sin\;\sin}_{\sin(\alpha_\ell x)\sin(\beta_p y)}\;\sinh\;dx\,dy$$

$$\delta_{\ell m}$$

$$\boxed{A_{mm} = \frac{4}{ab\sinh(\gamma_{mm} c)}\int_0^a dx\int_0^b dy\; V(x,y)\sin(\alpha_m x)\sin(\beta_m y)}$$

3.1 Laplace Eq. in Spherical Coordinates

$$\nabla^2 \Phi = 0, \quad \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

(r, θ, ϕ)

$$\Phi = \frac{U(r)}{r} P(\theta)\, Q(\phi)$$

Multiply by $\dfrac{r^3 \sin^2\theta}{UPQ}$

Last term becomes $\dfrac{1}{Q}\dfrac{d^2 Q(\phi)}{d\phi^2} = -m^2$

$\boxed{Q = e^{\pm i m \phi}}$; $\phi \to \phi + 2\pi \Rightarrow \boxed{m = \text{integer}}$

Divide by $\sin^2\theta$

$$\underbrace{\frac{r^2}{U(r)}\frac{d^2U(r)}{dr^2}}_{l(l+1)} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{-l(l+1)} = 0$$

$$\Rightarrow \frac{d^2U}{dr^2} - l(l+1)\frac{U}{r^2} = 0$$

$$\boxed{U = Ar^{l+1} + Br^{-l}}$$

$$U' = (l+1)Ar^{l} + B(-l)r^{-l-1}$$

$$U'' = (l+1)l\,Ar^{l-1} + B(-l)(-l-1)\,r^{-l-2}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

$x=\cos\theta$, $\sin^2\theta=1-x^2$

$$\frac{d}{d\theta}=\frac{d\cos\theta}{d\theta}\frac{d}{d\cos\theta}=-\sin\theta\frac{d}{d(\cos\theta)}=-\sqrt{1-x^2}\frac{d}{dx}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0$$

Generalized Legendre Equation

Read 97-100 Jackson

$\underline{m=0}$

$l = 0$ or a positive integer

Legendre polynomials

$P_0(x) = 1$

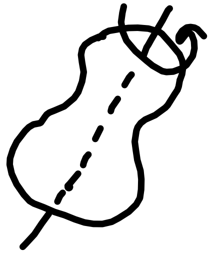
$P_1(x) = x$

$P_2(x) = \frac{(3x^2-1)}{2}$

$x = \cos\theta$

$dx = -\sin\theta \, d\theta$

$$\boxed{\int_{-1}^{1} P_{l'}(x) P_l(x)\, dx = \frac{2}{2l+1}\delta_{ll'}}$$

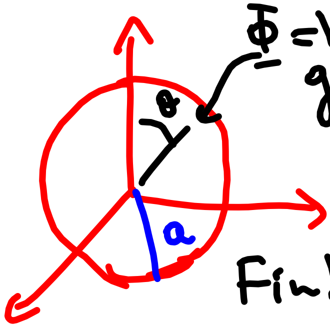


3.3 Problems with Azimuthal Symmetry

no ϕ dependence in the solution

$m = 0$

$$\Phi = \frac{U(r)}{r}P(\theta) = \sum_{l=0}^{\infty}\left(A_l r^{l} + B_l r^{-l-1}\right)P_l(\cos\theta)$$



$\Phi = V(\theta)$ given

Find Φ inside

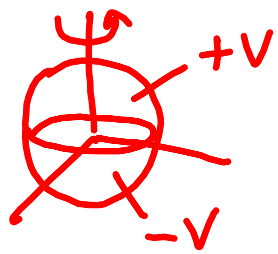
$B_l = 0$ to avoid divergences

$$V(\theta) = \sum_{l=0}^{\infty} A_l a^{l} P_l(\cos\theta)$$

$V(\theta)$: given; A_l: unknown

Multiply by $\sin\theta\, P_m(\cos\theta)$, integrate, use orthogonality relation.

$$\boxed{A_\ell = \frac{2\ell+1}{2a^\ell}\int_0^\pi V(\theta)P_\ell(\cos\theta)\sin\theta\, d\theta}$$



Solved book page 101

3.5 Spherical Harmonics

$m \neq 0$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

$\ell = 0, 1, 2, 3, \cdots$

$m = -\ell, -(\ell-1), \cdots, 0, \cdots, (\ell-1), \ell$

$P_\ell^m(\cos\theta) \leftarrow$ solutions

$$\underbrace{P_\ell^m(\cos\theta)\, e^{\pm im\phi}}$$

$$Y_{\ell m}(\theta,\phi) \overset{\text{def}}{=} \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

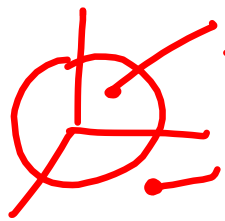
$$\boxed{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\; Y^*_{\ell' m'}(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}}$$

orthogonality

$$Y_{00} = \frac{1}{\sqrt{4\pi}}\;;\; Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\;;\; \ldots$$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \left(\underline{A_{\ell m}}\, r^{\ell} + \underline{B_{\ell m}}\, r^{-\ell-1} \right) Y_{\ell m}(\theta,\phi)$$

most general potential



Φ inside, $B_{\ell m} \equiv 0$

Φ outside, $A_{\ell m} \equiv 0$