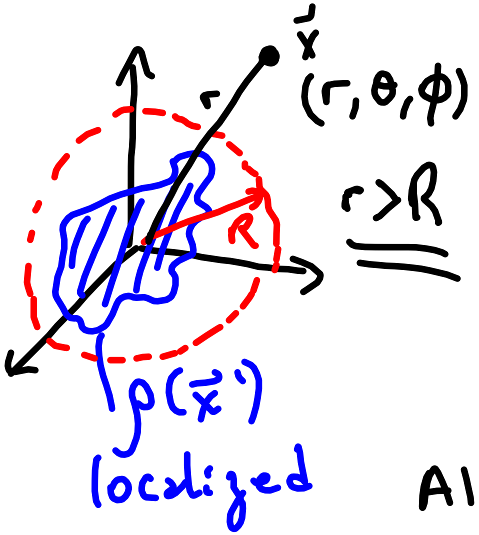
Mid term. Feb.13 given ; Feb.20 returned
No class Feb.18

4.1 Multipole expansion



$r > R$

$$\Phi(r,\theta,\phi) = \sum_{l,m} \underbrace{B_{lm}}_{\frac{1}{4\pi\epsilon_0}\frac{4\pi}{2l+1}\underbrace{q_{lm}}_{\text{to be found}}} r^{-l-1} Y_{lm}(\theta,\phi)$$

$l=0$ monopole
$l=1$ dipole
$l=2$ quadrupole
.....

Also

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\int_R d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Compare and find q_{lm}.

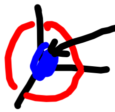
Magic formula:

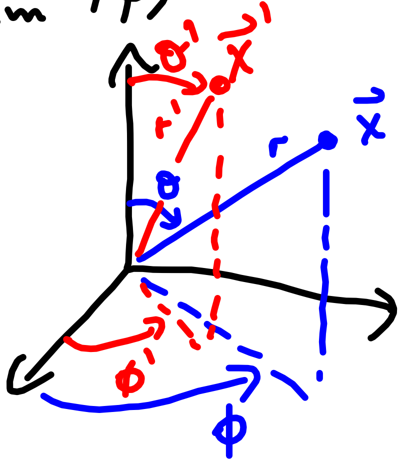
$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^{l}}{r^{l+1}} Y^{*}_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)$$

$r = |\vec{x}|$

$r' = |\vec{x}'|$

$r' < r$





$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \underbrace{\left(\int d^3x' \, \rho(\vec{x}')\, r'^{l}\, Y^{*}_{lm}(\theta',\phi') \right)}_{q_{lm}} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$$q_{00} = \int d^3x'\, \rho(\vec{x}')\, \underbrace{Y^*_{00}(\theta',\phi')}_{\to \frac{1}{\sqrt{4\pi}}}(r')^0 \leftarrow \ell=0 \quad = \frac{1}{\sqrt{4\pi}} \underbrace{\int d^3x'\, \rho(\vec{x}')}_{q} = \frac{q}{\sqrt{4\pi}}$$

$$q_{11} = \int \underbrace{Y^*_{11}(\theta',\phi')}_{-\sqrt{\frac{3}{8\pi}}\,\sin\theta'\,\underbrace{e^{-i\phi'}}_{(\cos\phi' - i\sin\phi')}}\, r'\, \rho(\vec{x}')\, d^3x'$$

$r'\sin\theta'\cos\phi' = x'$

$r'\sin\theta'\sin\phi' = y'$

$$= -\sqrt{\frac{3}{8\pi}} \int (x' - i y')\, \rho(\vec{x}')\, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}}\,(p_x - i p_y)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z$$

electric dipole moment

$$\vec{p} \stackrel{\text{def}}{=} \int \vec{r}'\, \rho(\vec{x}')\, d^3x'$$

$\uparrow (x', y', z')$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\vec{x}') (x'^2 - 2ix'y' - y'^2)$$

$\underline{\underline{l=2}}$

- - - - - - - -

Linear combination of q_{2m}

Quadrupole moment tensor

3x3 matrix

traceless

$r'^2 = x'^2 + y'^2 + z'^2$

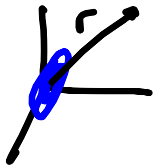
$$Q_{11} = \int d^3x' \rho(\vec{x}') (3x'^2 - r'^2)$$

$$Q_{12} = \int d^3x' \rho(\vec{x}')\, 3x'y'$$

- - - - - - - - - - - - - - - - -

$$\boxed{Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\vec{x}') d^3x' \qquad i,j = 1,2,3}$$

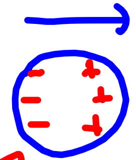
$$\boxed{\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \cdots\right]}$$



4.3 Electrostatic with Ponderable Media

Previous example

$\vec{E}_0$

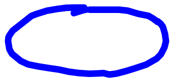


metallic sphere

electric dipole was induced

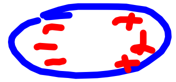
Molecules

$\vec{E}_0 = 0$

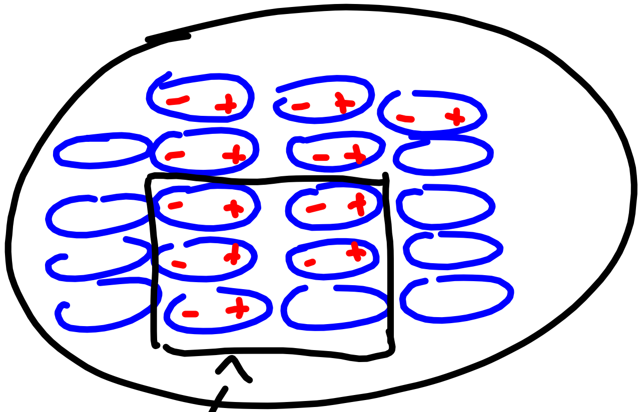


$\vec{p} = 0$

$\vec{E}_0 \neq 0$



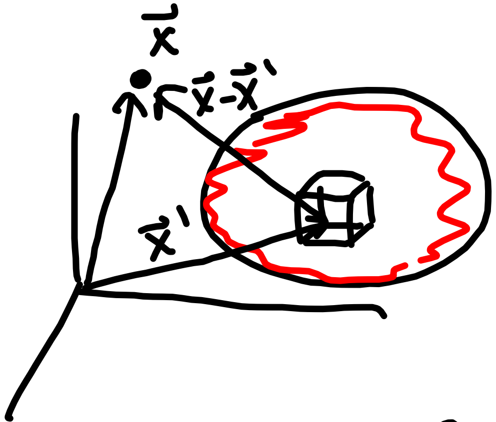
$\vec{p} \neq 0$



Φ

$\vec{P}(\vec{x})$ ← electric polarization

average
over a small volume
but much larger than unit cells



$$\Delta\Phi \text{ from } \square \approx \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(\vec{x}')\Delta V}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right]$$

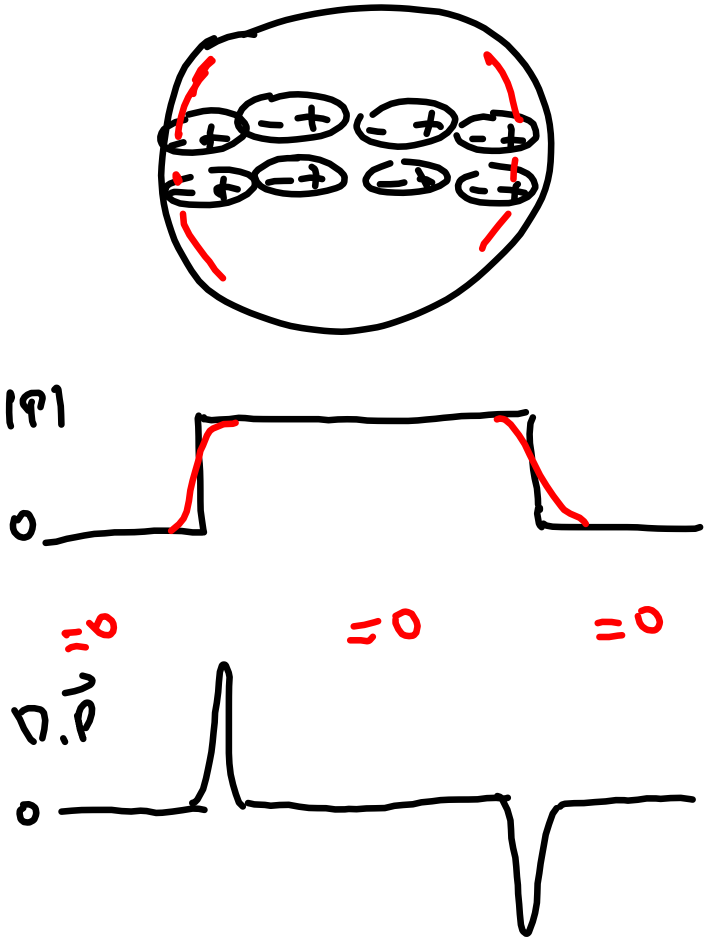
$$\Phi(\vec{x}) \approx \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\underbrace{\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}_{\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\underbrace{\left[\rho(\vec{x}') - \nabla\cdot\vec{P}(\vec{x}')\right]}_{\rho_{eff}(\vec{x}')}$$

$$\nabla\cdot(\psi\vec{F}) = \nabla\psi\cdot\vec{F} + \psi(\nabla\cdot\vec{F})$$

$$\psi = \frac{1}{|\vec{x}-\vec{x}'|};\ \vec{F} = \vec{P}$$

$$\int \nabla\cdot(\psi\vec{F}) = \int_S \psi\vec{F}\cdot d\vec{s} \to 0$$



$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \quad \text{vacuum}$$

$$\Downarrow$$

$$\nabla^2 \phi = -\frac{\rho_{eff}}{\epsilon_0} \quad \text{medium}$$

$$= -\frac{1}{\epsilon_0}\left(\rho - \nabla\cdot\vec{P}\right)$$

$$\boxed{\vec{E} = -\nabla\phi}$$

$$\boxed{\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho - \nabla\cdot\vec{P}\right)}$$

$\vec{D}$ (displacement vector)

$$\vec{D} \stackrel{\text{def}}{=} \epsilon_0 \vec{E} + \vec{P}$$

$$\nabla\cdot\vec{D} = \epsilon_0 \underbrace{\nabla\cdot\vec{E}}_{\frac{1}{\epsilon_0}(\rho - \cancel{\nabla\cdot\vec{P}})} + \cancel{\nabla\cdot\vec{P}} = \rho$$

$$\boxed{\nabla\cdot\vec{D} = \rho}$$

$$\boxed{\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho - \nabla\cdot\vec{P})}$$

Constitutive relation

$\vec{D} = f(\vec{E})$

often linear

$\vec{P} = \epsilon_0 \chi \vec{E}$

electric suscep. of the medium