## Review previous lecture

$$\nabla^2\phi = -\frac{1}{\epsilon_0}\underbrace{(\rho - \nabla\cdot\vec{P})}_{\rho_{eff}}$$

$$\nabla\times\vec{E} = 0$$

$$\vec{E} = -\nabla\phi$$

$$\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho - \nabla\cdot\vec{P})$$

$$\vec{D} \stackrel{def}{=} \epsilon_0\vec{E} + \vec{P}$$

$$\boxed{\nabla\cdot\vec{D} = \rho}$$

## Constitutive relation

$$\vec{D} = f(\vec{E})$$

linear response $\vec{E}_0$

$$\vec{P} = \epsilon_0\chi\vec{E}$$

$$\vec{D} = \epsilon_0\vec{E} + \epsilon_0\chi\vec{E} = \underbrace{\epsilon_0(1+\chi)}_{\epsilon}\vec{E}$$

$$\vec{D} = \epsilon\vec{E}$$

$\frac{\epsilon}{\epsilon_0}$ = dielectric constant $\epsilon(\vec{k},\omega)$

$\epsilon$ isotropic

$$\rho = \nabla\cdot\vec{D} = \nabla\cdot(\epsilon\vec{E}) = \epsilon\nabla\cdot\vec{E}$$

$$\boxed{\nabla\cdot\vec{E} = \frac{\rho}{\epsilon}}$$

# Boundary conditions

$\nabla \cdot \vec{D} = \rho$

$\int \nabla \cdot \vec{D} = \oint_S \vec{D} \cdot \vec{n}\, ds$

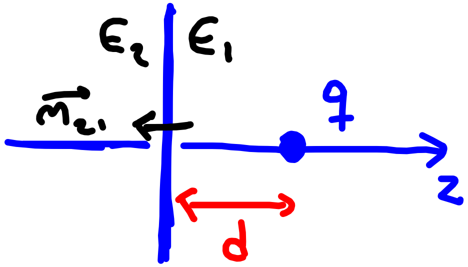
$$\boxed{(\vec{D}_2 - \vec{D}_1) \cdot \vec{n}_{21} = \sigma}$$

free charge

From $\nabla \times \vec{E} = 0 \longrightarrow$ $\boxed{(\vec{E}_2 - \vec{E}_1) \times \vec{n}_{12} = 0}$

# 4.4 Boundary problems with dielectrics. Example 1

## Method of images




Linear response. $\vec{D} = \epsilon \vec{E}$

$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_1}$ , $z > 0$

$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_2} = 0$ , $z < 0$

$\nabla \times \vec{E} = 0$ , $z<0$ and $z>0$

$(\vec{D}_2 - \vec{D}_1) \cdot \vec{n}_{21} = 0$

$D_{2z} - D_{1z} = 0$

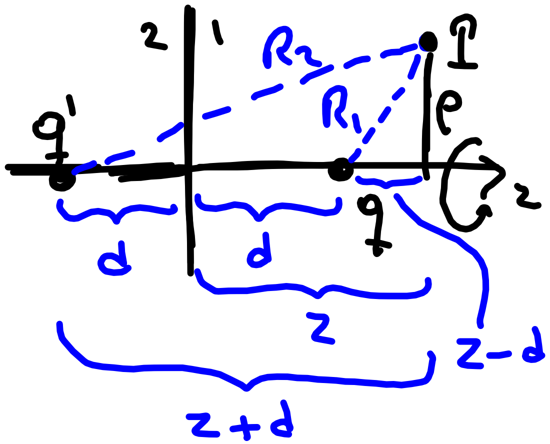
$\epsilon_2 E_{2z} - \epsilon_1 E_{1z} = 0$

z

$(\vec{E}_2 - \vec{E}_1) \times \vec{n}_{21} = 0$, $\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ (E_2 - E_1)_x & \dots & \dots \\ 0 & 0 & -1 \end{vmatrix} = 0$

$(E_2 - E_1)^y = 0$

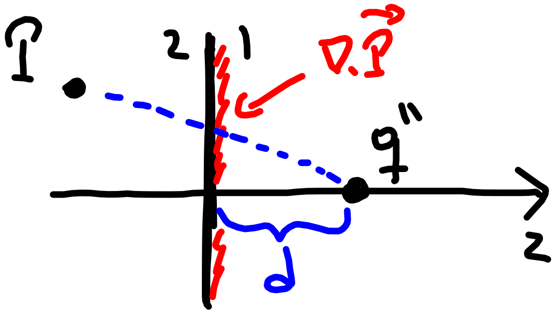
$(E_2 - E_1)^x = 0$




$$\Phi_{\text{at } P,\ z>0} = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right)$$

$$R_1 = \sqrt{\rho^2 + (z-d)^2}$$

$$R_2 = \sqrt{\rho^2 + (z+d)^2}$$




$$\Phi_{\text{at } P,\ z<0} = \frac{1}{4\pi\epsilon_2}\cdot\frac{q''}{R_1}$$

$q', q''$ unknowns to be found by BC

## From first B.C.

$$\epsilon_2 E_{2z} = \epsilon_1 E_{1z}$$

$$\vec{E} = -\nabla\phi$$

$$\left.\frac{d}{dz}\left(\frac{1}{R_1}\right)\right|_{z=0} = \frac{d}{(\rho^2+d^2)^{3/2}} \; ; \; \left.\frac{d}{dz}\left(\frac{1}{R_2}\right)\right|_{z=0} = \frac{-d}{(\rho^2+d^2)^{3/2}}$$

$$\left.\cancel{\epsilon_1}\frac{d}{dz}\left(\frac{1}{4\pi\cancel{\epsilon_1}}\left(\frac{q}{R_1}+\frac{q'}{R_2}\right)\right)\right|_{z=0} = \left.\cancel{\epsilon_2}\frac{q''}{4\pi\cancel{\epsilon_2}}\frac{d}{dz}\left(\frac{1}{R_1}\right)\right|_{z=0}$$

$$q - q' = q''$$

## Second B.C.

$\frac{d}{d\rho}\left(\frac{1}{R_1}\right), \frac{d}{d\rho}\left(\frac{1}{R_2}\right)$

$\boxed{\frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2}}$ ; $\boxed{q - q' = q''}$

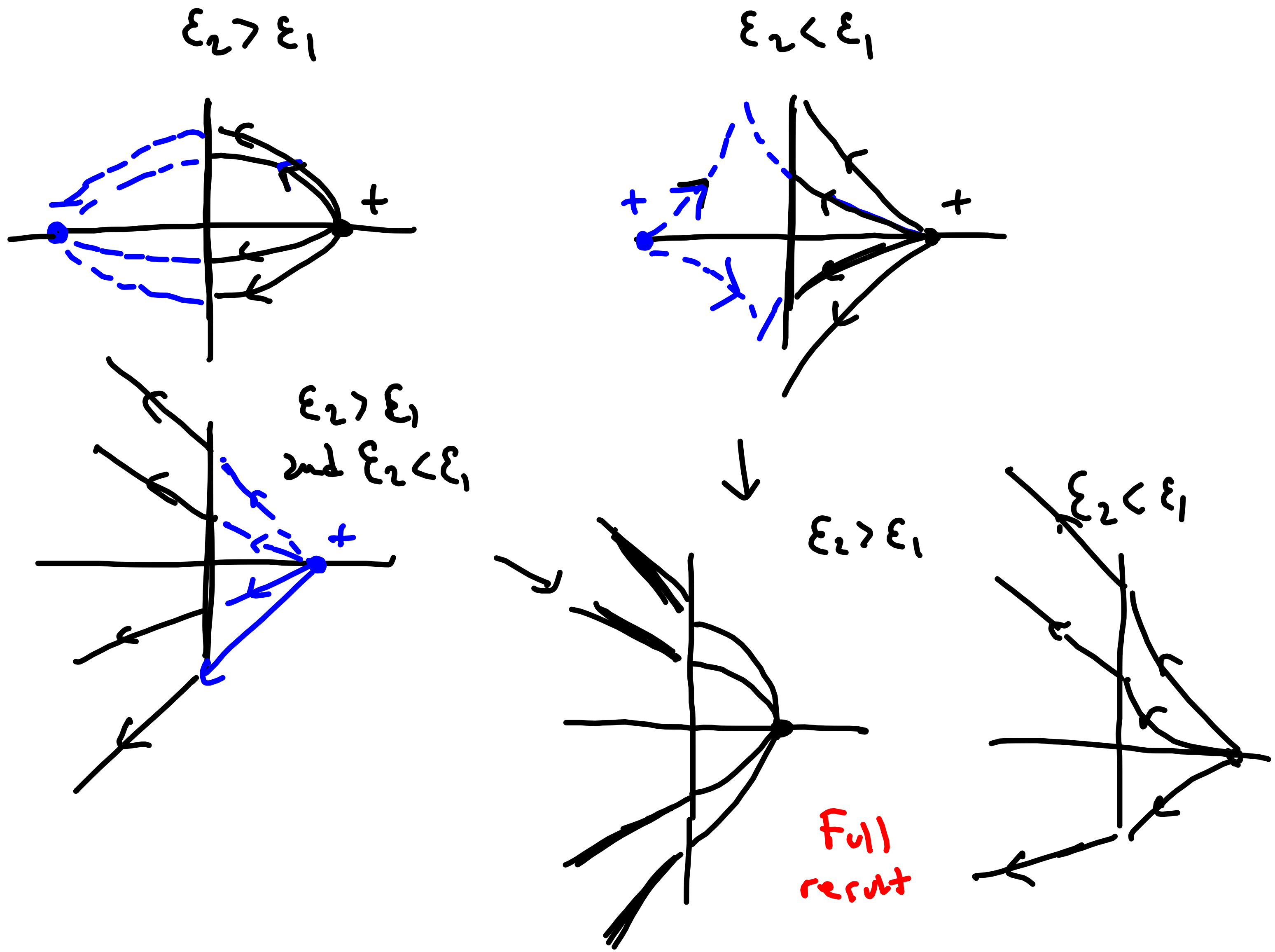
$\Downarrow$

$\boxed{q' = -\frac{(\epsilon_2-\epsilon_1)}{(\epsilon_2+\epsilon_1)} q}$ ; $\boxed{q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2} q}$

$\epsilon_2$ large

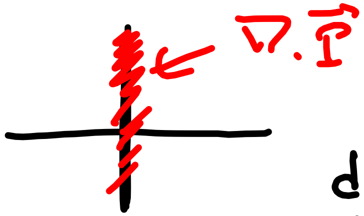
$\phi = \frac{1}{4\pi\epsilon_2}(\cdots\cdots)$ $q' \xrightarrow[\epsilon_2\to\infty]{} -q$

metal | $\cdot q$


$\varepsilon_2 > \varepsilon_1$
$\varepsilon_2 < \varepsilon_1$
$\varepsilon_2 > \varepsilon_1$ and $\varepsilon_2 < \varepsilon_1$
$\varepsilon_2 > \varepsilon_1$
$\varepsilon_2 < \varepsilon_1$
Full result

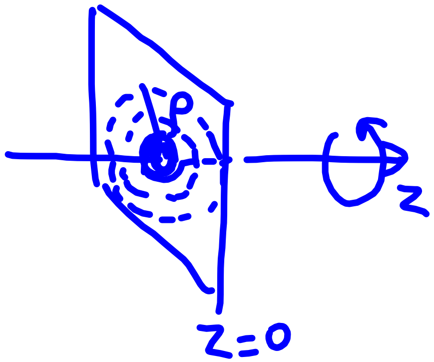
$$\rho_{eff} = \cancel{\rho}^{\,0} - \nabla\cdot\vec{P}$$

at interface




$$\frac{df(x)}{dx} \approx \frac{f(x+a) - f(x)}{a}$$

$$\Delta P_z = P^2_z - P^1_z \Big|_{z=0} =$$




$$= -\frac{q}{2\pi}\frac{\epsilon_0}{\epsilon_1}\frac{(\epsilon_2-\epsilon_1)}{(\epsilon_1+\epsilon_2)}\frac{d}{(\rho^2+d^2)^{3/2}} = \sigma_{pol}$$

$$\underset{\uparrow \atop \Delta E}{\sigma_{eff}} = \underset{\uparrow \atop \Delta D}{\cancel{\sigma}^{\,0}} - \underset{\uparrow \atop \Delta P}{\sigma_{pol}}$$

metal | $\epsilon_0$

$\epsilon_1 \to \epsilon_0$

$\epsilon_2 \to \infty$

$$\sigma = -\frac{q}{2\pi}\frac{d}{(\rho^2+d^2)^{3/2}}$$